Ricerche Economiche (1993) **47**, 281–291

Cointegration and aggregation

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Summary

This paper explores the conditions under which cointegration at the micro level implies cointegration at the macro level and vice versa. The aggregation conditions considered in this paper are in terms of common factors assumptions rather than the representative agent assumption, thereby allowing for a certain kind of heterogeneity among agents.

J.E.L. Classification: C32, C43.
Keywords: Aggregation, cointegration, common factors.

1. Introduction

Many economic theories are designed at the micro level and tested at the macro level. In a world of $I(1)$ variables, these economic theories will produce cointegration (by the transversality conditions) at the micro level, but this micro cointegration may not go through to the macro level. When this happens, rejection of macro cointegration does not automatically imply the rejection of a particular economic theory that has been designed for a representative agent. In fact, when cointegration is not preserved during the aggregation process, we will not even get consistent estimators of the macro parameters, making the macro model useless. One of the objectives of this paper is to show under which assumptions cointegration goes through the aggregation process, and then see which economic theories could be rejected or not using macro data.

Earlier literature suggests that the equality of the cointegrating vectors at the micro level is the only plausible condition for cointegration to be invariant under aggregation (see Lippi, 1988). This is not necessarily correct as the following simple example, with only two groups of agents and three variables, shows:

$$\begin{pmatrix} x_{1t} \\ y_{1t} \\ z_{1t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} f_t + I(0) \quad \text{and} \quad \begin{pmatrix} x_{2t} \\ y_{2t} \\ z_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} f_t + I(0), \qquad (1)$$

where f_t is $I(1)$. The cointegrating vectors at the micro level for

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group 1 and 2 are $\alpha_1=(1, -1, 1)$ and $\alpha_2=(1, -0{\cdot}5, -0{\cdot}25)$ respectively. These two micro cointegrating vectors are linearly independent and the aggregate variables are still cointegrated, as can be seen from their common factor representation,

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{pmatrix} f_t + I(0). \tag{2}$$

Notice that the macro cointegrating vector, $\alpha=(1, -2/3, 4/9)$, is not even a linear combination of the micro cointegrating vectors, α_1 and α_2.

This example shows that micro cointegration can go through to the macro level with heterogeneous individuals. In general, when the micro cointegrating vectors are not equal, what is needed for cointegration to be preserved in the aggregation process is "enough" extra cointegration in the full system formed by all the variables. In the above example this is ensured by the fact that f_t is the same for both groups of agents. Section 2 of this paper shows when the standard condition of the equality of the cointegrating vectors at the micro level is a necessary and sufficient condition in order to have cointegration at the macro level.

Most of the aggregation literature only considers the direction from micro to macro but not the other direction. Sometimes it is this second direction that is relevant from the empirical point of view. In many situations we can only rely on macro data. Micro data is either not available, or when it is, it contains measurement errors. In these cases it is important to study which kind of constraints the macro behaviour imposes on the micro behaviour, especially under which conditions macro cointegration implies micro cointegration.

Using the common factor representation that every cointegrated system has, it is straightforward to construct examples where the variables are cointegrated at one level of aggregation but not at the other.

This paper is organized as follows. Section 2 provides conditions under which cointegration at the micro level implies cointegration at the macro level. It also shows when the equality of the micro cointegrating vectors is necessary and sufficient for that implication to hold. Section 3 considers the other direction, presenting conditions for cointegration at the macro level to imply cointegration at the micro level. Section 4 is the conclusion. The proof of the theorems is given in the Appendix.

2. From micro to macro

Suppose that x_t is a series that has to be differenced (integer) d times to achieve a stationary series. x_t is then denoted $I(d)$ and thus a stationary series is $I(0)$. For simplicity only $d \leqslant 1$ will be considered here. A set of $I(1)$ variables $x_t = (x_{1t}, \ldots, x_{pt})$ is said to be cointegrated with rank r, if there exists a matrix $\underset{r \times p}{\alpha'}$ such that $\alpha' z_t$ is $I(0)$. This concept is discussed in Granger (1986) and in Engle and Granger (1991).

Suppose that there are n micro-units (groups of agents or commodities, regions, sectors of an economy,...) and that the same p variables are measured in each, giving the variables $x_{jk,t}\, j=1, \ldots, n$ and $k=1, \ldots, p$. Denote by $x_{j-,t} = (x_{j1,t}, \ldots, x_{jp,t})'$ the $p \times 1$ vector of variables for the j-th micro-unit. The aggregated variables are defined as

$$Sx_t = \sum_{j=1}^{n} x_{j-,t} = \left(\sum_{j=1}^{n} x_{j1,t}, \ldots, \sum_{j=1}^{n} x_{jp,t} \right)' = (S_1 x_t, \ldots, S_p x_t)'.$$

These aggregated variables, Sx_t, as well as the individual ones, $x_{j-,t}$, are assumed to be $I(1)$.

All the results in this paper are stated in terms of the following two representations of the vector $x_{j-,t}$: the Wold's representation,

$$\Delta x_{j-,t} = C_j(L)\varepsilon_{j-,t} = C_j(I)\varepsilon_{j-,t} + \Delta C_j^*(L)\varepsilon_{j-,t}, \tag{3}$$

where L is the lag operator and ε_t is a vector white noise process; and the factor representation,

$$\underset{p \times 1}{x_{j-,t}} = \underset{p \times s_j}{A_j} \; \underset{s_j \times 1}{f_{jt}} + I(0), \tag{4}$$

with f_{jt} a vector of non-cointegrated random walks. The vector $x_{j-,t}$ is cointegrated with rank r_j if and only if rank $\{C_j(I)\} = p - r_j$ or, equivalently, $s_j = p - r_j$.

Conditions for micro cointegration to imply macro cointegration involve constraints on A_j and/or f_{jt}. Many of these conditions do not have any clear meaning or interpretation. The next theorem presents a set of sufficient conditions that have a straightforward interpretation. These conditions depend on how much cointegration is in the system, and/or how similar the cointegrating vectors, α_j, are.

THEOREM 1: *Cointegration at the micro level implies cointegration at the macro level if any of the next three conditions are satisfied:*

(1) *The amount of cointegration (q) among the different micro-units is such that*

$$(n \times p) - \left(\sum_{j=1}^{n} r_j \right) - q < p. \tag{5}$$

(2) *The intersection of the null spaces spanned (SP) by A'_j is not empty,*

$$\bigcap_{j=1}^{n} SP\{N(A'_j)\} \neq \emptyset \tag{6}$$

(3) *A mixture of conditions (1) and (2).*

This theorem is a generalization of proposition $2-a$ in Gonzalo (1989) and of fact 2 in Lippi (1989).

Condition (1) can be stated in two different equivalent ways:

(1′) The number of unit roots or common $I(1)$ factors in the full system is less than p,

(1″) the variables in the j-th micro-unit admit the factor representation

$$x_{j-,t} = P_j S x_t + I(0), \tag{7}$$

where the rank $(P_j) = r_j$.

Notice that condition (1) is automatically satisfied when there are more variables than micro-units, $p > n$, and the rank of cointegration in every micro-unit is $p-1$. A much more interesting case is when all the $I(1)$ common factors are the same.

$$\underset{s \times 1}{f_{jt} = f_t}.$$

In this situation the amount of cointegration in the whole system is $R = n(p-s) + (ns-s)$ and because s is always less than p, condition (1), $(n \times p) - R < p$, is satisfied.

An example of equal common $I(1)$ is found in those consumption theories (see Deaton, 1992) where everyone (y_j) gets a share of the aggregate income (y), together with an idiosyncratic and transitory shock,

$$\Delta y_{jt} = \lambda_j \varepsilon_t + u_{jt} - u_{jt-1} \text{ and } \Delta y_t = \varepsilon_t. \tag{8}$$

A simpler version of (8) is

$$y_{jt} = \lambda_j y_t, \quad \text{with} \quad \sum_{j=1}^{n} \lambda_j = 1. \tag{9}$$

Following with the consumption–income example, we conclude that cointegration of these two variables at the individual level implies cointegration between total consumption and total income, if the individual incomes are cointegrated with rank $n-1$, no matter what the individual marginal propensities to consume are.

In a more general framework, we can assume, as Blanchard and Quah (1989), that there are two kinds of shocks in the economy: some that have a permanent effect (maybe supply shocks) in the system and others that only have a transitory effect (maybe demand shocks). Condition (1) holds if the number of permanent shocks in the whole system is less than the number of variables in each micro-unit.

One implication of condition (1), and in general of this paper, is that the search for common $I(1)$ factors of the full system becomes very important, to see if we can aggregate in terms of cointegration. Approximations to these common factors can be found from the error correction model of the appropriated system, as discussed in Stock and Watson (1988) and Gonzalo and Granger (1991).

Condition (2) says that if there exists at least one cointegrating vector that is equal for all the micro-units, then cointegration goes through the aggregation process. This condition was initially stated in Lippi (1989), and the next theorem shows when this is the only plausible way to get cointegration at the aggregate level.

THEOREM 2: *The existence of at least one common cointegration vector in all the micro-units is a necessary and sufficient condition for micro-cointegration to imply macro-cointegration if any of the next three conditions are satisfied:*

(4) *The vector* $(\varepsilon_{1-,t}, \ldots, \varepsilon_{n-,t})$ *is a vector white noise process with full rank covariance matrix.*
(5) *The long-run covariance* (Ψ) *of* $(\varepsilon_{1-,t}, \ldots, \varepsilon_{n-,t})$ *has full rank.*
(6) *The common factors* $f_{1t}, \ldots, f_{n,t}$ *in (4) are not cointegrated.*

Notice that condition (4) implies condition (5) and (5) and (6) are equivalent.

Many models commonly used in the simultaneous equation panel data literature (see Hsiao, 1986) satisfy the above conditions. For instance, it can be proved that the following specification,

$$\varepsilon_{jk,t} = \eta_{jt} + \mu_{jt} + \beta_{kt} + e_{jk,t}, \qquad j=1, \ldots, n, \quad k=1, \ldots, p, \tag{10}$$

where all the components are independent of each other, uncorrelated across time and

$$E(\eta_t \eta_s)=\sigma^2_\eta \text{ if } t=s, \text{ 0 otherwise;}$$
$$E(\mu_{jt}\mu_{it})=\sigma^2_{\mu j} \text{ if } j=1, \text{ 0 otherwise;}$$
$$E(\beta_{kt}\beta_{mt})=\sigma^2_{\beta k} \text{ if } k=m, \text{ 0 otherwise;}$$
$$E(e_{jkt}e_{imt})=\sigma^2_{ejk} \text{ if } (j,k)=(i,m), \text{ 0 otherwise;}$$

satisfies conditions (4), (5) and (6). A particular case of (10) is when $\varepsilon_{j-,t}$ are independent across micro-units. In this case the covariance matrix Ψ is block diagonal, and because the $\varepsilon_{j-,t}$ are non-degenerate white noise processes, it has full rank.

Going back to the consumption–income example, if the model for the individual incomes, instead of (8), is

$$\Delta y_{jt}=\lambda_j\varepsilon_t+u_{jt}-\theta_j u_{jt-1},\ \theta_j\neq 1, \tag{11}$$

with $(u_{1t},\ldots,u_{nt})$ a vector white noise, then by Theorem 2, condition (6), cointegration between consumption and income at the individual level implies cointegration between total consumption and total income if, and only if, all the individuals have the same marginal propensity to consume. Therefore, if the process for the individual incomes is (11), and at least two consumers have different marginal propensity to consume, we will never get cointegration at the aggregate level. It is in this sense that Theorem 2 could be interpreted as an impossibility result for aggregation.

Another interpretation of Theorem 2, in the same impossibility vein, is that if any of conditions (4), (5) or (6) are satisfied, then it is enough to have only one micro-unit that is not cointegrated in order for there not to be cointegration at the macro level.

A case where the conditions of Theorem 2 are not satisfied is example (1). This is why, in that example, there is micro and macro cointegration with non-equal micro cointegrating vectors.

3. From macro to micro

Adding up the individual Wold and factor representations, (3) and (4), we get the equivalent representations for the aggregate Sx_t:

$$\Delta Sx_t=C_1(I)\varepsilon_{1,t}+\ldots.+C_n(I)\varepsilon_{n,t}+(I-L)I(0) \tag{12}$$

and

$$Sx_t=A_1f_{1t}+\ldots.+A_nf_{nt}+I(0). \tag{13}$$

The vector Sx_t is cointegrated if and only if there exists an $\underset{r\times p}{\alpha'}$ such that

$$\alpha' C_1(I)\varepsilon_{1-,t}+\ldots.+\alpha' C_n(I)\varepsilon_{n-,t}=(I-L)I(0) \tag{14}$$

or, equivalently, if and only if

$$\alpha' A_1 f_{1t}+\ldots..+\alpha' A_n f_{nt}=I(0). \tag{15}$$

Cointegration at the macro level imposes certain constraints on the micro behaviour, as Granger (1992) shows, but they are not strong enough to ensure micro cointegration. For that we need to impose some extra conditions. Some of them are discussed in this section.

THEOREM 3: *Cointegration at the macro level (with rank r) implies cointegration at the micro level if any of the conditions (4), (5) or (6) of Theorem 2 are satisfied. In this case, there are r cointegration vectors common to all the micro-units.*

Everything that has been said about Theorem 2 applies here.

There are situations where the long-run covariance matrix (Ψ) of $\varepsilon_{1-,t},\ldots,\varepsilon_{n-,t}$) does not have full rank, and macro cointegration still implies micro cointegration. These are situations where there is enough cointegration in the system to ensure that all the $I(1)$ness comes from a number of common factors less than the number of variables in each micro-unit. This is summarized in the next theorem.

THEOREM 4: *Let $\tilde{q}=R-r$ be the number of cointegrating vectors in the whole system that are linearly independent of the macro cointegrating vectors. Then, macro cointegration with rank r implies micro cointegration if*

$$(n\times p)-r-\tilde{q}<p. \tag{16}$$

This theorem is the counterpart of condition (1) in Theorem 1. The reason it has been stated here again is to emphasize that in going from the macro level to the micro level the conditions are much stronger than for the other direction. The $\tilde{q}$ in (16) in general will be much bigger than the q in (5).

An interesting subcase of this theorem is when the extra cointegration ($\tilde{q}$) comes specifically from the variable space ($k=1,\ldots,p$). Denote $x_{\_k,t}=(x_{1k,t},\ldots,\chi_{nk,t})'$, the $n\times 1$ vector of micro-units for the

k-th variable, and suppose this vector has rank of cointegration r_k, then it admits the following factor representation:

$$\underset{n\times 1}{x_{-k,t}} = \underset{n\times s_k}{B_k} \underset{s_k\times 1}{g_{kt}} + \mathrm{I}(0), \tag{17}$$

with $s_k = n - r_k$ and $g_{kt} \sim I(1)$.

COROLLARY 1: *Macro cointegration (with rank r) implies micro cointegration if*

$$(n \times p) - r - \sum_{k=1}^{p} r_k < p \tag{18}$$

Notice that this corollary is automatically satisfied if $r_k = n - 1$, and it is not satisfied if $r_k = n - 2$. So, in order to get micro cointegration from macro cointegration we need a large amount of cointegration in the variable space across micro-units.

In the consumption–income example, what Corollary 1 requires is that the income of all individuals is cointegrated, and also that their consumption is cointegrated, with rank $n-1$. In Section 2, from micro to macro, only cointegration of the individual incomes was required.

4. Conclusions

This paper shows that to preserve cointegration in the aggregation (disaggregation) process, the representative agent assumption, used in most of the macro theories, is not necessary. We can still allow for certain heterogeneity among the agents or sectors, although not too much. What is required is that the $I(1)$ness of the system come from a small number of non-cointegrated $I(1)$ factors, or that all the micro-units share at least one cointegrating vector. These conditions are easily tested (see Johansen, 1988; Stock & Watson, 1988; Gonzalo & Granger, 1991). Whether they are still too strong to be satisfied in practice or not is something that has to be analysed empirically, case by case.

Acknowledgements

I thank Clive Granger, James Davidson, Rob Engle, Doc Ghose, Carlo Giannini and Megan Werner for helpful comments and discussions. I also thank the Bank of Spain and the Ramon Areces

Foundation for financial support during the first version of this paper, UCSD Working Paper, 89–30.

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Appendix

PROOF OF THEOREM 1: Condition (1) implies that there exists the following factor representation for the full system:

$$(x'_{1-,t}, \ldots, x'_{n-,t})' = \underset{p\times s}{(H'_1, \ldots, H'_n)'} \underset{p\times s}{} \underset{s\times 1}{w_t} + I(0), \tag{A.1}$$

where $s = (n \times p) - \left(\sum_{j=1}^{p} r_j\right) - q < p$, and w_t is a vector of non-cointegrated random walks. Adding up (A.1), we get

$$Sx_t = \sum_{j=1}^{p} x_{j-,t} = \left(\sum_{j=1}^{p} \underset{p\times s}{H_j}\right) w_t + I(0). \tag{A.2}$$

So there exists an $\underset{r \times p}{\alpha'}$ such that

$$\underset{r \times p}{\alpha'} \Big(\sum_{j=1}^{p} \underset{p \times s}{H_j} \Big) = 0, \quad r = p - s \tag{A.3}$$

and therefore Sx_t is cointegrated.

Condition (2) implies that there exists a vector $\underset{r \times p}{\alpha'}$ such that

$$\alpha' A_j = 0, j = 1, \ldots, n. \tag{A.4}$$

Therefore, from (15), Sx_t is cointegrated,

$$\alpha' Sx_t = 0 + \alpha' I(0) = I(0). \tag{A.5}$$

Q.E.D.

Proof of Theorem 2: It is sufficient to prove this theorem only for condition (6).

Micro cointegration, (4), implies macro cointegration if there exists an $\underset{r \times p}{\alpha'}$ such that

$$\alpha' A_1 \underset{s_1 \times 1}{f_{1t}} + \ldots + \alpha' A_n \underset{s_n \times 1}{f_{nt}} \text{ is } I(0) \tag{A.6}$$

where $s_j < p, j = 1, \ldots, n$. If $(f_{1t}, \ldots, f_{nt})$ are not cointegrated, the only way (A.6) holds is if (A.4) is satisfied. This implies that all the micro-units share r cointegrating vectors.

Q.E.D.

Proof of Theorem 3: It is sufficient to prove this theorem only for condition (5).

The vector Sx_t is cointegrated if and only if

$$\alpha' C_1(I) \varepsilon_{1-,t} + \ldots + \alpha' C_n(I) \varepsilon_{n-,t} = (1-L) I(0). \tag{A.7}$$

The fact that $\varepsilon_{j-,t}$ is a white noise process does not imply that the whole vector $(\varepsilon_{1-,t}, \ldots, \varepsilon_{n-,t})$ is a vector white noise. But it is always possible to find a vector white noise process u_t such that

$$(\varepsilon'_{1-,t}, \ldots, \varepsilon'_{n-,t})' = \varphi(L) u_t. \tag{A.8}$$

The long-run covariance matrix of $(\varepsilon'_{1-,t}, \ldots, \varepsilon'_{n-,t})'$ is

$$\Psi = \varphi(I) \operatorname{Var}(u_t) \varphi(I)'. \tag{A.9}$$

Multiplying (A.7) by $(\varepsilon'_{1,t}, \ldots, \varepsilon'_{n,t})'$ and taking expectations in both sides, at frequency zero, we get

$$(\alpha' C_1(I), \ldots, \alpha' C_n(I))\, \Psi = 0 \tag{A.10}$$

Therefore, if Ψ has full rank, (A.10) holds if and only if (A.4) is satisfied. This implies that all the micro-units share r cointegrating vectors.

Q.E.D.

PROOF OF THEOREM 4: See proof of Theorem 1.

Q.E.D.

PROOF OF COROLLARY 1: From (17) we can write

$$x_{j-,t} = \tilde{A}_j(g'_{1t}, \ldots, g'_{pt})' + I(0), \tag{A.11}$$

where $\dim(\tilde{A}_j) = p \times \{\sum_{k=1}^{p} (n - r_k)\}$ and $\dim(g_{kt}) = n - r_k$. Adding up (A.11), we get

$$Sx_t = \sum_{j=1}^{n} x_{j-,t} = (\tilde{A}_1 + \ldots + \tilde{A}_n)(g'_{1t}, \ldots, g'_{pt})' + I(0). \tag{A.12}$$

Macro cointegration (with rank r) implies

$$(g'_{1t}, \ldots, g'_{pt})' = \underset{s \times 1}{F w_t} + I(0) \tag{A.13}$$

with $s = \sum_{k=1}^{p} (n - r_k) - r$. Then, if $s < p$, from (A.11) we get cointegration at the micro level.

Q.E.D.