DYNAMIC CONTESTS WITH BANKRUPTCY: THE DESPAIR EFFECT*

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Abstract

In this paper we analyze a two period contest in which agents may become bankrupt at the end of the first period. A bankrupt agent is excluded from the contest in the second period of the game. We investigate the existence of a subgame perfect equilibrium in pure strategies. We distinguish between a borrowing equilibrium where at least an agent might be bankrupt and a non borrowing equilibrium where no agent is bankrupted. We prove that the former occurs when the agent taking loans is relatively poor. This is the despair effect where severely handicapped agents take actions that are risky. We also show conditions under which both kind of equilibria overlap or not. We provide an example in which no equilibrium exists.

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1. Introduction

The theory of contests studies conflicts where agents spend effort in order to obtain a prize. In static contests, aggregate effort is maximized when agents are identical. So if I play tennis against Nadal, I will make very little effort -because my chances to win are very small- and Nadal will make very little effort too because he does not need much effort to defeat me. In dynamic contests this translates into the *Discouragement Effect* where lagging players make little effort or throw the towel -because they will loose with a high probability- and players with a large advantage will do little effort too because they need not much to win, see Konrad (2012) and the references therein.

However, in some dynamic contests, the contrary may well happen. Suppose that in the Champions League, a soccer team after losing in the first half of the first round is offered a drug that will increase performance in the second half but, with same probability the drug will be discovered and the team disqualified. If the losing team is defeated by, say 1-0, it might reject the offer on the grounds that it is too risky. But if the team is losing 3-0 it might accept the drug because the chances that it can overturn the result without extra help are slim. The result will be more effort in the second half of the game but perhaps less effort in the long run because if the illegal drug is discovered the team will be disqualified. Thus heterogeneity in players may well increase aggregate effort, at least in some periods. We call this situation the *Despair Effect* because handicapped players may find optimal to take risky actions that will not be sensible if these players were not handicapped.

In this paper, we present a two period complete information contest in which two agents are endowed with money and they can get extra money in a capital market.\(^1\) Potential lenders can either invest in the safe asset or in a contestant. The latter is risky because if this contestant does not win, investors get no return. The capital market equalizes expected returns of both assets. Thus, agents with shallow pockets may overcome this handicap by raising loans money and competing in more equal terms with agents with deep pockets. Examples of this situation are wars among empires for a resource and repeated competition among firms for public (i.e. aircraft for US navy) or private (building construction) procurement.\(^2\)

\(^1\)See Brander and Lewis (1988) for a study of the role of financial constraints in duopoly.

\(^2\)The initial motivation for this research came from a sentence of Sir Norman Foster in the film *How much does your building weigh, Mr. Foster?* about the dire straits suffered by his studio before they were awarded the HSBC Hong Kong building.
A crucial assumption of our model is that a contestant unable to repay the loan will be excluded in the second period contest. This assumption is an idealization of the problems faced by a country or a firm unable to repay its debts. It has been backed by Bolton and Scharfstein (1990) in a strategic finance setup and has been used by Eaton and Gersovitz (1981) in their analysis of sovereign debt. In the next section we will discuss this assumption. For the time being let us recall Mr. Micawber’s famous, and often quoted, recipe for happiness:

"Annual income twenty pounds, annual expenditure nineteen [pounds] nineteen [shillings] and six [pence], result happiness. Annual income twenty pounds, annual expenditure twenty pounds ought and six, result misery."

This is explained by the fact that, as it turns out in the novel, the slightest debt will put the debtor in jail, no matter how small. Another example of our assumption is the recent UEFA proposal to enforce budget balance among all clubs playing European competitions. To play there, clubs must prove they have no outstanding payments to players, to each other or to the tax authorities.

We distinguish between two scenarios. In the first, one of the agents (the rich agent) has very large money endowments so he never takes a loan and never faces risk of liquidation. The other agent (the poor agent) has limited money endowment, he might get a loan so he either wins the contest or face liquidation. We call this scenario Rich Man-Poor Man. We prove that the pure strategy subgame perfect Nash equilibrium (SPNE) of this game is unique and can be of two, mutually exclusive, types:

1. The poor agent finances entirely his expenses with his endowments. We call this the non borrowing equilibrium.

2. The poor agent finances part of his expenses in the capital market. We call this the borrowing equilibrium.

In the non borrowing equilibrium both agents spend less than in the standard one shot Nash equilibrium (NE). In the borrowing equilibrium both agents spend more than they do in the one shot NE. The latter can be seen as a kind of predatory equilibrium where the rich agent spends a large quantity of money which drags the poor agent into borrowing which in turn, given the risk of bankruptcy of the latter, increases the expected prize received by the rich in the second period.
The (non) borrowing equilibrium exists when the poor agent is (resp. is not) very poor and he does not care (resp. he cares) much about the future. The role of discount is clear: If an agent does not care much about the future, the risk of bankruptcy has small payoff consequences so he is inclined to get a loan. The role of the endowment is precisely the despair effect. It comes from the fact that a loan allows the poor agent to compete on equal footing with the rich agent. When both agents have similar money endowments, a loan does not mean much to the poor agent, in terms of helping him to compete in the first period, and brings a risky outcome. We also show that for intermediate values of the poor’s endowment and the discount rate a SPNE may not exist.

In the second scenario, both agents are identical, have limited endowments and may access the capital market. We call this scenario Poor Man-Poor Man. This scenario is meant to capture the polar situation to the one in the previous section and to see the impact of the deep pocket in equilibrium outcome. Now we have two equilibria one in which both agents do not borrow and another in which both agents borrow. We show that these equilibria have properties that closely match those encountered in the Rich Man-Poor Man scenario. The only difference is that both equilibria can coexists. Thus the impact of the size of pockets appears to be technical in nature, namely about the existence and uniqueness of equilibria but not about the properties of the latter.

Our paper is related with other papers in which the result of early rounds may encourage contestants to make more effort. Sela (2011) considers a race in which the loser cares about the magnitude of the defeat and shows that the loser of the first battle may be encouraged to increase effort in the second battle to avoid a dishonorable defeat. Beviá and Corchón (2013) consider a two period contest when the strength in the second contest depends on the result of the first contest. Thus, winning in the first round has an impact on the outcome in the second round so players have an extra incentive to spend effort. Consequently, the discouragement effect holds only when the difference between players is sufficiently large. Garfinkel and Skaperdas (2000) study the effect of war on pacification in subsequent periods. The despair effect considered in this paper refers to cases in which effort today might bring disastrous consequences in the future. Examples of this effect abound in the military history from the dictum "caja o faja" (coffin or belt, a military regalia only wore by marshals), which refers to low rank officers commanding almost suicidal attacks that in case of success will bring big promotions, to battles like Leite Gulf in 1944 in which the Japanese navy committed almost all available ships to defend crucial oil supply lines to Japan. The famous dictum
in the "Communist Manifesto" (1848) that "The proletarians have nothing to lose but their chains. They have a world to win" may be interpreted as another example even though, understandably, Marx and Engels did not emphasize the dire consequences on proletarians of defeat.

The rest of the paper goes as follows: section 2 describes the model and states some preliminary results. The first and the second scenarios previously described are analyzed in Sections 3 and 4 respectively.

2. The Model

There are two periods and two agents (also called contestants). In each period, say $t$, agents contest for a prize of value $V$ by spending a quantity of a resource that we call money and denote by $G_i^t$ where $i \in \{1, 2\}$ denotes the agent. In period one agent $i$ is endowed with $M_i$ units of money. Without loss of generality we assume that $M_1 \geq M_2$. If $G_1^1 > M_1$ agent $i$ can borrow from a credit market where money can be invested either in financing the contestants or in a riskless asset which after a period yields $r$ units per unit investment. The interest $r$ is determined exogenously. An investment of a unit of money in the expenses made by contestant $i$ yields $s$ with probability $p_i$ and 0 with probability $1 - p_i$. Thus, assuming that investors are risk neutral, the expected return is $p_is$. If the capital market is competitive we should have that

$$p_is = r.$$  \hfill (2.1)

Thus if the "risky" investment is a safe deal, $p_i \equiv 1$ and then $s \equiv r$. And when the risky investment is very risky, $p_i \equiv 0$ and then $s \equiv \infty$.

Let us now write the expected payoffs of contestant $i$ who fully financed $G_i^1$ units of money through the capital market and spent them in the contest. With probability $p_i$ it wins $V$ but it has to pay $sG_i^1$. With probability $1 - p_i$ it loses and it has no money to pay. Thus expected profits for contestant $i$ are

$$p_iV - p_isG_i^1 = p_iV - rG_i^1.$$  \hfill (2.2)

If the expenses are financed with the endowments, they have an opportunity cost of $r$. Thus, in any case, equation (2.2) represents the payoff of contestant $i$. In the rest of the paper without loss of generality we set $r = 1$.

In each period there are three stages defined as follows:
1. **Agents decide the amount of expenses.**

   If this amount exceeds the available money, they borrow the difference, i.e. if agent $i$ spends $G^t_i > M_i$ he can borrow $G^t_i - M_i$ in the credit market. We emphasize that the decisions on expenditure and borrowing are simultaneous, see our comments later.

2. **The prize is awarded.**

   Let $p^t_i$ be the probability that agent $i$ obtains the price in period $t$. In each period the probability of winning the contest is given by a Contest Success Function (CSF) written as

   $$ p^t_i = p_i(G^t_i, G^t_j) \tag{2.3} $$

   where $p_i(\cdot)$ is twice continuously differentiable in $\mathbb{R}^2_{++}$, $p_i(0, G^t_j) = 0$ for $G^t_i > 0$, $\partial^2 p_i(G^t_i, G^t_j)/\partial G^t_i G^t_j < 0$, strictly increasing in $G^t_i$ and

   $$ \frac{\partial^2 p_i(G^t_i, G^t_j)}{\partial G^t_i G^t_j} > 0 \iff G^t_i > G^t_j. \tag{2.4} $$

   We also assume that the CSF is symmetric, that is $p_i(x, y) = p_j(y, x)$, homogeneous of degree zero, i.e. the units in which expenses are measured do not affect the result. Our final assumption is a kind of Inada condition, namely that $\partial p_i(y, y)/G^t_i$ tends to zero when $y$ goes to infinity, and it tends to a number greater than $1/V$ when $y$ goes to zero. An example of a CSF satisfying all these conditions is the Tullock CSF, namely

   $$ p_i(G^t_i, G^t_j) = \frac{G^t_i}{G^t_i + G^t_j}, i, j \in \{1, 2\}, i \neq j. \tag{2.5} $$

3. **Bankruptcy rules.** If the agent was in debt and did not win the prize, no one wants to lend him anymore so he is excluded from the contest in period 2.

   This assumption is, of course, an idealization. Couwenberg (2001) finds that the survival rate of firms after bankruptcy is 18% US, 20% in UK and 6% in France. And countries may fall into several bankruptcies before they are out of the world domination game: Spanish Habsburgs became bankrupt in 1557, 1576, 1596 and 1607 before the bankruptcies that sealed his fate in 1647 and 1653. France, bankrupt at the eve of French revolution in 1789, enjoyed an enviable position in European affairs until 1813. But in both cases, bankruptcies had serious
consequences on the role of these nations.\textsuperscript{3} Even near bankruptcies, like Scotland in 1707, Great Britain in 1945 and Russia in 1990’s paved the way for the reduced visibility of these nations in subsequent years. In fact Paul Kennedy (1987) argues that financial overburden, caused by overexpansion in strategic commitments, is the main cause of the decline of empires. We take this view to the limit assuming that bankrupted nations disappear from the contest arena. In the conclusions we retake this discussion and propose a more general set up.

We say that an agent is active in the second period if he can participate in the contest. If an agent loses the contest in the first period, he will be active in the second period iff $G_i^1 \leq M_i$. If an agent wins the contest in the first period he will be active in the second period iff $G_i^1 \leq V + M_i$.

Finally we assume that:

4 \textbf{Second period}. If in this period there is only one active agent, this agent wins the prize at no cost. If there are no active agents, the prize is not awarded. If two agents are active they compete like in the first period. If an agent cannot repay the loan, this has no consequences because the world ends in this period. Therefore, in the second period if both agents are active they spend the same money and obtain the same payoff which we denote by $\pi$.

In period $t$, expected payoff of agent $i$ is

$$\pi_i^t = p_i(G_i^t; G_j^t)V - G_i^t.$$ \hfill (2.6)

Expected payoff for agent $i$ for the whole game is denoted by $\pi_i$ and defined as

$$\pi_i = \pi_i^1 + \delta \pi_i^2.$$ \hfill (2.7)

where $\delta \in (0, 1]$ is the discount rate, common to both agents.

Note that bankruptcy in the second period does not have consequences on the exclusion of agents since there are no more contests to play. Thus, all the action occurs in period one and consequently we focus our analysis in this period. Our equilibrium concept is subgame perfect Nash equilibrium (\textit{SPNE}).

\textsuperscript{3}Cruces and Trebesch (2012) construct a database of investor losses in all restructurings of sovereign debt from 1970 until 2010, covering 180 cases in 68 countries and find that "high creditor losses are associated with... longer periods of market exclusion."
In the Appendix 1 we gather some technical implications of the assumptions above that will be used later on in the proofs of the main results.

In the next section we focus on the case where agent 1 can pay out of his endowments any conceivable expense but agent 2 cannot. We will call this case "Rich Man - Poor Man". The case in which both agents are constrained ("Poor Man-Poor Man") will be analyzed in the subsequent section.\(^4\)

3. Rich Man - Poor Man Scenario

In this section we assume that agent 1 -the rich agent- has a very large quantity of money so he will never be constrained, and agent 2 -the poor agent- has not.

Consider a game in which payoff functions are \(\pi_1^1\) and \(\pi_2^1\) (see 2.6) and there is no financial constraints. Let the best reply function of player \(i\) be denoted by \(RO_i(G_j)\). We call the Nash equilibrium of this game the one shot Nash equilibrium. Under our assumptions this equilibrium exists, is unique and symmetric (see Appendix 1). Let \(\tilde{G}\) be the expense of an agent in the one shot Nash equilibrium. For future reference let \(\bar{\pi}\) be the one shot Nash equilibrium payoffs.

We assume that the poor agent cannot finance \(\tilde{G}\) out of his pocket, i.e. \(M_2 < \tilde{G}\). Consequently the poor agent has to decide if he wants to borrow or not. This case is analytically convenient as a start because it simplifies the handling of the financial constraints. It correspond to the "deep pocket" case which has been considered in oligopolistic markets, see Bolton and Scharfstein (1990) and the references therein. It has been argued that deep pockets yield predation. We will consider the validity of such a conclusion in our set up.

There are two possible kinds of equilibria of the dynamic game: those in which the poor agent does not borrow -equilibrium without borrowing- and those in which the poor agent borrows -equilibrium with borrowing.

We start with the case in which the poor agent does not borrow in equilibrium. One would expect that this equilibrium exists when the money endowments of the poor agent are large (but smaller than \(\tilde{G}\), of course). Our first proposition shows that this conjecture is true.

\(^4\)In the two remaining cases i.e. that agent 1 is constrained but agent 2 is not and that both agents are unconstrained- the first is impossible and in the second the SPNE consists in the repetition of the actions that are a Nash equilibrium in the one shot game.
**Proposition 1.** Given $\delta \in (0,1)$ there exists $M_2^{NB}(\delta) \in (0,\bar{G})$ such that an equilibrium without borrowing exist if and only if $M_2 \geq M_2^{NB}(\delta)$.

**Proof.** See Appendix 2.

The proof establishes that when $M_2$ tends to zero, borrowing is better than no borrowing because with the latter option the poor agent has no chance of winning. Conversely when $M_2$ is close to $\bar{G}$ borrowing makes little difference and it implies a risk so borrowing is not a good option. The intermediate value theorem tells us the existence of a point in which both options are indifferent. And since poor man payoffs are increasing with $M_2$, the result follows.

We note the following:

**Remark 1.** In a non-borrowing equilibrium both agents spend less than if they were unconstrained.

For the poor agent this is by definition and for the rich agent it follows from (2.4). Thus hard financial constraints make all agents spend less because they occur only when strategies are strategic complements.

A different question would be to consider a given endowment of the poor agent and see if there is a $\delta$, say $\delta$, such that an equilibrium without borrowing exists for all $\delta \in (\delta,1)$. This would be reasonable because for $\delta \approx 1$ agents care dearly about the consequences of a possible bankruptcy in the first period so they prefer the safe option of not borrowing. However this concern may be not enough to deter the poor agent from deviating from the non-borrowing situation as shown in the following example.

**Example 1.** Assume that the CSF is given by (2.5). The equilibrium in the one shot game is given by:

$$G_1 = G_2 = \bar{\pi} = \frac{V}{4}$$

(3.1)

Note first that if agent 2 is constrained, $M_2 < V/4$. To show the existence of a non borrowing equilibrium we have to show that $G_1^3 = M_2$ and $G_1^1 = RO_1(M_2) = \sqrt{VM_2} - M_2$ is an equilibrium. The payoff of the poor agent if he does not borrow is:

$$\pi_2^{NB} = \frac{M_2}{M_2 + G_1^1}V - M_2 + \frac{\delta V}{4} = \sqrt{M_2V} - M_2 + \frac{\delta V}{4} = G_1^1 + \frac{\delta V}{4}.$$  

(3.2)
If he deviates and decides to borrow, he will risk bankruptcy but will increase his probability of winning in the first period. Then his continuation payoff is

$$\pi_2^B = p_2V - \hat{G}_2 + \delta(p_2\frac{V}{4}) = p_2(V + \delta\frac{V}{4}) - \hat{G}_2$$

(3.3)

where $\hat{G}_2 = \sqrt[4]{V(1 + \frac{1}{4}\delta)G_1^1 - G_1^1}$, and $p_2 = \frac{\hat{G}_2}{G_2 + G_1^1}$. (3.4)

A non borrowing equilibrium exists if and only if the payoffs when the poor agent borrows are smaller than the payoffs when he does not borrow, namely

$$\sqrt{VM_2} - M_2 \geq \frac{V}{4(1 + \frac{1}{4}\delta)}$$

(3.5)

By setting $q \equiv \frac{V}{M_2}$ the necessary and sufficient condition (3.5) can be written as:

$$\sqrt{q} - 1 \geq \frac{q}{4(1 + \frac{1}{4}\delta)}$$

(3.6)

Recall that since agent 2 is constrained, $M_2 < V/4$, that is, $q > 4$. If $\delta = 1$ (which is the most favorable case for the existence of a non borrowing equilibrium), equation (3.6) implies that $q$ must be smaller than 13.09, that is, the initial wealth of the poor agent should be at least 7% of the value of the prize. Thus, when the poor agent is indeed very poor the strategy of borrowing and risking bankruptcy pays off.

Note that (3.6) defines $q$ as an increasing function of $\delta$. Thus, under a Tullock CSF, non borrowing equilibrium arises as a combination of patient agents and the poor agent not being very poor.

We now turn our attention to equilibrium with borrowing. Firstly we note that, given $M_2$, when $\delta \simeq 0$ an equilibrium with borrowing exists because the poor agent can enhance his chances of winning the first period contest by borrowing and he does not care about the possible consequences of the second period. So let us work out the converse: one would expect that for a given $\delta$ when the poor agent is indeed very poor he will choose to borrow regardless of the probability of bankruptcy. Our next proposition shows that, under an additional assumption, this is indeed the case.

Let $\hat{G}_1^1, \hat{G}_2^1$ be the Nash equilibrium if the poor agent were forced to borrow. This equilibrium exists under our assumptions, see Appendix 2. Let $\pi_2^B(\hat{G}_1^1, \hat{G}_2^1)$ be poor player payoffs in such equilibrium.

**Proposition 2.** Let $\delta \in (0, 1)$ be such that $\pi_2^B(\hat{G}_1^1, \hat{G}_2^1) > \delta\pi$. Then, there exists $M_2^B(\delta) \in (0, G)$ such that for all $M_2 \leq M_2^B(\delta)$ an equilibrium with borrowing exists. If $\pi_2^B(\hat{G}_1^1, \hat{G}_2^1) < \delta\pi$ no
equilibrium with borrowing exists. In a borrowing equilibrium the rich agent will make more effort than the poor agent.

**Proof.** See Appendix 2.

The strategy of the proof of Proposition 2 is akin to Proposition 1. The result on efforts follows from the fact that the payoff of the rich agent is larger when the poor agent has a possibility of being bankrupt. Consequently the rich agent is more aggressive in the first period. This is the "Shadow Effect" where the anticipation of a future weaker contender makes strong players more aggressive (see Brown and Minor (2011)).

In Proposition 2 we prove the existence of an equilibrium with borrowing given that \( \pi_2^B(\tilde{G}_1, \tilde{G}_2) > \delta \pi \). Notice that, given \( \tilde{G}_1 \), by not borrowing, agent 2 can always guarantee himself at least \( \delta \pi \) (expending nothing in the first period will give him this payoff). Thus, if by borrowing he get less than that, an equilibrium with borrowing will not exist. With a Tullock CSF for instance, if \( \delta = 1 \), the condition \( \pi_2^B(\tilde{G}_1, \tilde{G}_2) > \pi \) does not hold.

In order to further understand this issue let us again work out the case for the Tullock CSF.

**Example 2.** Assume that the CSF is given by (2.5). If an equilibrium with borrowing exists, expenses in the first period are given by:

\[
G_1^1 = \sqrt{V(1 + \frac{3}{4}\delta)G_2^1 - G_2^1}, \quad \text{and} \quad G_2^1 = \sqrt{V(1 + \frac{1}{4}\delta)G_1^1 - G_1^1}. \tag{3.7}
\]

Solving (3.7), we get that

\[
G_1^1 = \frac{(1 + \frac{3}{4}\delta)^2(1 + \frac{1}{4}\delta)}{(2 + \delta)^2}V, \quad \text{and} \quad G_2^1 = \frac{(1 + \frac{3}{4}\delta)(1 + \frac{1}{4}\delta)^2}{(2 + \delta)^2}V. \tag{3.8}
\]

Both \( G_1^1 \) and \( G_2^1 \) are increasing in \( \delta \). Since for \( \delta = 0 \), \( G_1^1 = V/4 > M_2 \), agent 2 is borrowing. The prize in case of borrowing for the rich agent is bigger because with certain probability he will be the only one surviving in the second period and thus, \( G_1^1 > G_2^1 \).

The probabilities for each player of winning the contest in the first period are:

\[
p_1 = \frac{(1 + \frac{3}{4}\delta)}{(2 + \delta)}, \quad \text{and} \quad p_2 = \frac{(1 + \frac{1}{4}\delta)}{(2 + \delta)}. \tag{3.9}
\]

Thus, the payoff for the poor agent if he borrows is:

\[
\pi_2^B = p_2 V - G_2^1 + \delta(p_2 \frac{V}{4}) = \frac{(1 + \frac{1}{4}\delta)^3}{(2 + \delta)^2} V. \tag{3.10}
\]
For \((G_1, G_2)\) to be an equilibrium we need to checked first that \(RO_2(G_1) > M_2\), because otherwise given \(G_1\) agent 2 can best reply without borrowing and risking bankruptcy. And secondly, we have to show that the poor agent does not have incentives to deviate and play safe, that is, not to borrow and play \(M_2\). Since \(RO_2(G_1) = \sqrt{V G_1} - G_1\), the condition \(RO_2(G_1) > M_2\) can be written as:

\[
V \frac{(1 + \frac{3}{4} \delta)}{(2 + \delta)} \sqrt{(1 + \frac{1}{4} \delta)} - \frac{(1 + \frac{3}{4} \delta)^2 (1 + \frac{1}{4} \delta)}{(2 + \delta)^2} V > M_2.
\] (3.11)

Dividing by \(M_2\) and letting \(q = V/M_2\) as before we get

\[
q \left(\frac{(1 + \frac{3}{4} \delta)}{(2 + \delta)} \sqrt{(1 + \frac{1}{4} \delta)} - \frac{(1 + \frac{3}{4} \delta)^2 (1 + \frac{1}{4} \delta)}{(2 + \delta)^2}\right) > 1.
\] (3.12)

Given (3.12), let us see that agent 2 does not have incentives to deviate. If he plays \(M_2\) given that agent 1 is playing \(G_1\), his payoff will be:

\[
\pi_2^{NB} = \frac{M_2}{G_1 + M_2} V - M_2 + \frac{V}{4}.
\] (3.13)

Thus, he will not deviate if \(\pi_2^{NB} < \pi_2^B\) or equivalently

\[
\frac{1}{\frac{(1 + 3 \delta)^2 (1 + \frac{1}{4} \delta)}{(2 + \delta)^2} q + 1} q - 1 + \frac{\delta}{4} q < \frac{(1 + \frac{1}{4} \delta)^3}{(2 + \delta)^2} q.
\] (3.14)

Condition (3.14) says that, for a given \(q\), when the poor agent does not care much about the future (i.e. \(\delta\) is close to 0), he may well risk bankruptcy. Alternatively, given \(\delta\), the poorer the agent is relatively to \(V\) (i.e. \(q\) is high), the more likely he prefers the risky strategy of borrowing resources. Thus, the borrowing equilibrium arises as a combination of impatient agents and the poor agent having really small endowments. Another interesting feature of this equilibrium is that the expenses of both players are larger than those in the one shot equilibrium. Thus, in this equilibrium agent 1 challenges agent 2 with a large expense and agent 2 accepts the challenge. Finally note that when \(q\) is very large (\(M_2\) small) an equilibrium with borrowing exists iff \((\delta/4) < (1 + \frac{1}{4} \delta)^3/(2 + \delta)^2\) which does not hold for \(\delta\) close to 1. Thus, without the condition that \(\pi_2^B(G_1, G_2) > \delta \pi\) an equilibrium with borrowing does not exists.

In Figure 1, the area below the dotted line (which is very close to the \(\delta\) axis) corresponds to condition (3.12) and the area below the solid line corresponds to condition (3.14). Thus, for any \((q, \delta)\) in the area below the solid line an equilibrium with borrowing exist (recall that the poor agent is constrained only when \(q > 4\)). The area above the dash line (3.6) corresponds to the space of parameters \((q, \delta)\) for which an equilibrium without borrowing exists.
It is easy to check the following remark

**Remark 2.** Assume that the CSF is given by (2.5). Then equilibrium is unique.

To end this section note that for any pair \((q, \delta)\) between both lines in Figure 1 there is no a SPNE in pure strategies. In Appendix 3 we present an example of non existence.

Summing up, in the rich man-poor man case equilibrium can be of two kinds: either the poor agent accepts his fate and spends less than if financial constraints would not exist (equilibrium without borrowing) or he risks bankruptcy (equilibrium with borrowing). The first equilibrium occurs when concerns for the future are important and the poor agent is not very poor. The second equilibrium exist in the opposite circumstances and, when the CSF is Tullock, is characterized by both agents spending more than in the one shot equilibrium. This is because the rich agents has incentives to force the poor agent to accept a large risk of bankruptcy. Thus, this is kind of predatory equilibrium like when US forced USSR in the early eighties to a large military expenditure which accelerated the demise of the socialist state. For intermediate values of concern about the future and wealth of the poor agent, an equilibrium might not exists.
4. Poor Man - Poor Man Scenario

In this section we study the case where no agent can pay out of his endowments the expenses corresponding to the one shot game, \( M_1 < \bar{G} \) and \( M_2 < \bar{G} \). To keep things simple we focus here in the case in which both agents have identical endowments, thus \( M_1 = M_2 \). This case is somehow the polar case to the one considered in the previous section, where the asymmetry arising from endowments was maximal.

In the rest of this section we analyze the two possible equilibria.

4.1. Non Borrowing Equilibrium

Here we prove a result akin to Proposition 1 in the previous section.

**Proposition 3.** If both agents have identical endowments, given \( \delta \in (0, 1) \) there exists \( M^{NB}(\delta) \in (0, \bar{G}) \) such that an equilibrium without borrowing exist if and only if \( M \geq M^{NB}(\delta) \).

The proof is shown in Appendix 4. Again the result hinges on the fact that when \( M \) is relatively large borrowing is a risky option that brings little good and conversely when \( M \) is small borrowing is the best way to compete.

As in the previous section, we work out in detail the case in which the CSF is Tullock.

**Example 3.** In a non borrowing equilibrium, given that agents are identically constrained, both agents expend their entire resources in the first period. Both agents survive in the second period, therefore their continuation payoff is identical for both of them and equal to \( \delta V/4 \). Thus,

\[
\pi_i^{NB}(M, M) = \frac{1}{2} V - M + \frac{\delta V}{4}, i \in \{1, 2\}. \tag{4.1}
\]

If agent \( i \) deviates and borrows, his continuation payoff changes once he faces bankruptcy with some probability. Thus, by playing \( G_i > M \), \( i \)'s payoff is

\[
\pi_i^B(G_i, M) = \frac{G_i}{G_i + M} V(1 + \frac{\delta}{4}) - G_i. \tag{4.2}
\]

The most profitable deviation will be to play the best reply of \( i \) against \( M \), that is \( G_i = \sqrt{V(1 + \frac{\delta}{4})M - M} \), given to agent \( i \) a payoff

\[
\pi_i^B(G_i, M) = \frac{\sqrt{V(1 + \frac{\delta}{4})M - M}}{\sqrt{V(1 + \frac{\delta}{4})M}} V(1 + \frac{\delta}{4}) - \sqrt{V(1 + \frac{\delta}{4})M} + M. \tag{4.3}
\]
An equilibrium with non borrowing will exist if and only if

\[ \pi^\text{NB}_i(M, M) \geq \pi^B_i(G_i, M), \quad (4.4) \]

The above inequality can be written as

\[ \frac{1}{2} - \frac{1}{q} \geq 1 - 2\sqrt{\frac{1}{q}(1 + \frac{\delta}{4}) + \frac{1}{q}}. \quad (4.5) \]

Summing up, and equilibrium with non borrowing exists if and only if (4.5) holds.

4.2. Equilibrium with borrowing.

Suppose that both agents borrow. In this case both are facing a probability of bankruptcy in the second period. Therefore, with probability \( p_i^1 \) agent \( i \) will be the only one surviving in the game and his continuation payoff will be \( \delta p_i^1 V \). At \( (G_1^1, G_2^1) \) with \( G_i^1 > M \), payoff for agent \( i \) is

\[ \pi^B_i(G_1^1, G_2^1) = p_i^1 V - G_i^1 + \delta p_i^1 V = p_i^1 (1 + \delta) V - G_i^1. \quad (4.6) \]

Now we prove a result similar to Proposition 2 in the previous section.

**Proposition 4.** Suppose both agents have identical endowments. Let \( \delta \) be such that \( \pi_i(\hat{G}, \hat{G}) \geq \delta \bar{\pi} \). There exists \( M^B(\delta) \in (0, \bar{G}) \) such that an equilibrium without borrowing exist if and only if \( M \leq M^B(\delta) \).

The proof is in Appendix 3. The intuition is as in Proposition 2.

As before, to get intuition, let us assume that the CSF is Tullock.

**Example 4.** Now \( (\hat{G}_1^1, \hat{G}_2^1) \) is

\[ \hat{G}_1^1 = \hat{G}_2^1 = \hat{G} = \frac{(1 + \delta)V}{4}, \quad (4.7) \]

is an equilibrium with borrowing.

If agent \( i \) deviates and does not borrow, that is, plays \( G_i^1 \leq M \), he will not face bankruptcy and with probability \( p_i(G_i^1, \hat{G}) \) he will be the only one surviving in the game and he will get \( V \) and, with probability \( 1 - p_i(G_i^1, \hat{G}) \), both agents will survive and he will get \( V/4 \). Thus, if he deviates by playing \( G_i^1 \leq M \), his payoff will be:

\[
\pi^\text{NB}_i(G_i^1, \hat{G}) = p_i(G_i^1, \hat{G}) V - G_i^1 + \delta(p_i(G_i^1, \hat{G}) V + (1 - p_i(G_i^1, \hat{G})) V/4) = \\
= p_i(G_i^1, \hat{G}) V(1 + \frac{3}{4} \delta) - G_i^1 + \delta V/4 \quad (4.8)
\]
First of all, let us see that by best replying according to $\pi^B_i$ to $\hat{G}$ without taking into account the constraints, let us say $\hat{G}_i^1$, is bigger than $M$. Thus, the best possible deviation will be to play $M$. Note that $\hat{G}_i^1 = \sqrt{(1 + \frac{3}{4}\delta)V\hat{G} - \hat{G}}$. Using the value of $\hat{G}$ and that $V/M = q$, we get that

$$\sqrt{(1 + \frac{3}{4}\delta)V\hat{G}_i^1 - \hat{G}_i^1} > M,$$

(4.9)

is equivalent to

$$\sqrt{(1 + \frac{3}{4}\delta)(1 + \delta) - \frac{(1 + \delta)}{4}} > \frac{1}{q}.$$  

(4.10)

Since $q > 4$, and the left hand side of (4.10) is increasing in $\delta$, the smallest value of the left hand side is $1/4$. Thus, condition (4.10) always holds. Therefore, we only need to see under what conditions deviating by non borrowing and playing $M$ is not profitable. If agent $i$ deviates and plays $M$ he gets

$$\pi^B_i(M, \hat{G}) = \frac{M}{(1 + \delta)V + M}V(1 + \frac{3}{4}\delta) - M + \delta V/4.$$  

Thus, the deviation is not profitable if and only if

$$\pi^B_i(M, \hat{G}) \geq \pi^B_i(M, \hat{G}).$$

That is,

$$\frac{(1 + \delta)V}{4} \geq \frac{M}{(1 + \delta)V + M}V(1 + \frac{3}{4}\delta) - M + \delta V/4,$$

(4.12)

or equivalently,

$$\frac{1}{4} \geq \frac{(1 + \frac{3}{4}\delta)}{(1 + \delta)q + 1} - \frac{1}{q}.$$  

(4.13)

Summing up, and equilibrium with borrowing exists if and only if (4.13) holds.

In Figure 2, for any $(q, \delta)$ in the area below the solid line an equilibrium with borrowing exist, and above the dash line an equilibrium without borrowing exist. The intuitions are identical to those supplied in the Rich Man-Poor Man scenario. Note that both equilibria can coexist, see Appendix 5 for a numerical example.
5. Conclusions and further extensions

In this paper we have presented a model of a two period contest where agents have money endowments and may borrow money. We assume that inability to repay the loan carries the disappearance of this agent. We have shown that relatively poor agents might take loans. Thus handicapped agents may take actions that endanger their survival in the long run but which, if successful, reduce substantially the handicap. We have called this the *Despair Effect* and we have shown that it exists in two polar scenarios: Rich Man-Poor Man where an agent has unlimited endowments and Poor Man-Poor Man where both agents are identical and have relatively small endowments.

Many questions remain to understand fully the *Despair Effect*. A natural extension, especially when the interpretation is that contestants are firms, is that agents compete an undetermined number of times for a prize of value $V$. Now firms have the possibility of colluding by, for instance, coordinating such that only one firm shows up in every contest. Suppose firm 1 shows up in odd periods and firm 2 shows up in even periods. Thus discounted profits are respectively $V/(1 - \delta)^2$ and $\delta V/(1 - \delta)^2$. If a firm breaks the collusive agreement wins in this period but the other firm will play non-cooperatively in the rest of the game. Assume a Tullock CSF. To start with the simplest case assume that both firms have zero endowments (i.e. a limit case of the Poor Man-Poor Man scenario) and that money cannot be transferred between periods. If a firm breaks the agreement
today wins \( V \) and if both firms will play non-cooperatively tomorrow in the second period they win in expected terms \( V/4 \). But since one of the firms is bankrupted in this period with probability \( 1/2 \) the deviator wins \( V/2 \) in expected terms in the remaining game. Thus the expected payoff for a deviator is \( V + \delta V/4 + V\delta^2/2 \). Collusion is an equilibrium iff

\[
\frac{\delta V}{(1 - \delta)^2} \geq V + \frac{\delta V}{4(1 - \delta)} + \frac{V\delta^2}{2(1 - \delta)},
\]

or equivalently

\[
\delta - (1 - \delta + \frac{\delta}{4} + \frac{\delta^2}{2})(1 + \delta) \geq 0,
\]

which is impossible. This is due to the fact that from the second period on, a firm is a monopolist (with probability .5) so breaking the collusive agreement yields a big reward. This example shows that the introduction of financial constraints changes completely the picture and, in this case, makes collusion impossible if only trigger strategies are used.

Finally we assumed two agents, a very specific bankruptcy rule and a stylized capital market. All these assumptions raise issues that must be considered in further research.

6. Appendix 1: Auxiliary Results

In this section we gather results that we will use in the proof of Propositions 1 and 2.

In what follows we drop the time superindex whenever does not create confusion.

If the contest were played once and agents were not constrained by money endowments, the best reply function of agent \( i \) would be given by the first order condition

\[
\frac{\partial p_i(G_{ij}, G_j)}{G_i} V - 1 = 0.
\]

We denote this best reply function by \( RO_i(G_j) \). For instance if the CSF is given by (2.5)

\[
RO_i(G_j) = \sqrt[4]{V G_j} - G_j.
\]

Totally differentiating (6.1) and using (2.4),

\[
\frac{dRO_i(G_j)}{dG_j} > 0 \Leftrightarrow G_i > G_j.
\]

Thus the best reply in the unconstrained one shot game is first increasing (strategic complementarity) and then decreasing (strategic substitution). Our assumptions on \( p_i(.) \) imply that this is also the case for the general \( p_i(.) \).
A one shot Nash Equilibrium (NE) is a pair \((G_i, G_j)\) such that \(G_i \in RO_i(G_j)\), \(i, j \in \{1, 2\}\), \(i \neq j\). Under our assumptions, a one shot NE exists, is unique and symmetric, i.e. \(G_i = G_j = \tilde{G}\) (see for example Proposition 1 in Corchón (2000)). Also each agent wins the contest with probability 1/2 and both agents receive the same payoff which, as we said at the end of point 4 above, we denote by \(\pi\). From (6.3) in such equilibrium it must be that

\[
\frac{dRO_i(\tilde{G})}{dG_j} = \frac{dRO_j(\tilde{G})}{dG_i} = 0. \tag{6.4}
\]

Property (6.4) plus (2.4) imply the following:

\[
\text{for all } (G_1, G_2) < (\tilde{G}, G_j), \ G_j < RO_i(G_j). \tag{6.5}
\]

This is because by defining \(\Psi(G_j) = RO_i(G_j) - G_j\), we see that \(\Psi(\tilde{G}) = 0\) and \(\Psi'(\tilde{G}) = -1\). Thus, for \(G_j\) close enough but smaller than \(\tilde{G}\), \(\Psi(G_j) > 0\). Thus by (2.4) again \(\Psi(G_j) > 0\) for all \(G_j < \tilde{G}\).

For future reference we note that, if the prize were valuated differently by each contestant, say in \(V_1\) and \(V_2\) units respectively, in the one shot Nash equilibrium, \((G_1^*, G_2^*)\), we will have \(V_1 G_2^* = V_2 G_1^*\) by homogeneity of degree zero. Thus, in equilibrium

\[
G_1^* > G_2^* \Leftrightarrow V_1 > V_2, \tag{6.6}
\]

which given (2.4) implies that

\[
G_1^* > G_2^* \Leftrightarrow \left\{ \frac{\partial^2 p_i(G_1^*, G_2^*)}{\partial G_1 \partial G_2} > 0, \frac{\partial^2 p_2(G_1^*, G_2^*)}{\partial G_1 \partial G_2} < 0 \right\}. \tag{6.7}
\]

The player with the larger (resp. smaller) valuation regards strategies as strategic complements (resp. substitutes).

In the sequel we will compare the best reply of an agent in two different games which only differ in the agents’ value of the prize. In this case, the best reply of agent \(i\) to \(G_j\) when the value of the prize for agent \(i\) is \(V_i\) is denoted by \(RO_i(G_j; V_i)\). Note that

\[
\text{if } V_i > V_i' \text{ then } RO_i(G_j; V_i) > RO_i(G_j; V_i'). \tag{6.8}
\]

This follows directly from the first order conditions and the assumption \(\partial^2 p_i(G_i, G_j)/\partial G_i^2 < 0\).

Finally, we assume that

\[
\lim_{G_i \to 0} p_i(G_i, RO_i(G_j; V_i)) = 0. \tag{6.9}
\]

Assumption (6.9) holds for Tullock CSF.

7.1. Preliminaries

Let us study the best reply of agent 1 in the first period. We will refer to this best reply as \( RD_1(G_2^1) \).

If the poor agent does not borrow, \( G_2^1 \leq M_2 \), both agents are active in the second period, they will get \( \bar{\pi} \) in the second period, and thus the best reply of agent 1 will be \( RO_1(G_2^1) \).

However, if the poor agent decides to borrow, his continuation payoff will be different from the non-borrowing case because with probability \( p_2 \) he will be active in the second period and with probability \( p_1 \) he will be bankrupt. Thus, the expected payoff for agent 1 will be:

\[
\pi_1^B = p_1 V - G_1^1 + \delta(p_2 \bar{\pi} + p_1 V) = p_1((1 + \delta)V - \delta \bar{\pi}) - G_1^1 + \delta \bar{\pi}. \tag{7.1}
\]

Note that this two period game is equivalent to a one shot game in which the prize for the rich agent is \((1 + \delta)V - \delta \bar{\pi}\) which is larger than \(V\).

The best reply of agent 1 is now given by the first order condition:

\[
\frac{\partial p_1(G_1^1, G_2^1)}{\partial G_1^1}((1 + \delta)V - \delta \bar{\pi}) = 1. \tag{7.2}
\]

We denote this best reply by \( RD_1^B(G_2^1) \) where \( B \) refers to the fact that the poor agent is borrowing.

Summarizing, the dynamic best reply for agent 1 is:

\[
RD_1(G_2^1) = \begin{cases} 
RO_1(G_2^1) & \text{if } G_2^1 \leq M_2 \\
RD_1^B(G_2^1) & \text{if } G_2^1 > M_2
\end{cases}. \tag{7.3}
\]

For the poor agent, the constraint agent, if \( G_1^1 \) is such that \( RO_2(G_1^1) \leq M_2 \) then, \( G_2^1 = RO_2(G_1^1) \). Otherwise, he will decide to borrow or not depending on the expected gains from borrowing. If he borrows, his expected payoff will be

\[
\pi_2^B = p_2 V - G_2^1 + \delta p_2 \bar{\pi} = p_2(V + \delta \bar{\pi}) - G_2^1, \tag{7.4}
\]

where \( G_2^1 \) is the unique solution of the first order condition

\[
\frac{\partial p_2(G_1^1, G_2^1)}{\partial G_2^1}(V + \delta \bar{\pi}) = 1.
\]

We denote this best reply by \( RD_2^B(G_1^1) \). In this case, again, the two period game is equivalent to a one shot game in which the prize for the poor agent is \( V + \delta \bar{\pi} \). Note that this prize is less
than the prize for agent 1, \(((1 + \delta)V - \delta\bar{\pi})\). This implies that agent 1 will make, in a borrowing equilibrium, more effort than the poor agent. The payoff of the rich agent is larger when the poor agent has a possibility of being bankrupt. Consequently the rich agent is more aggressive in the first period. This is the "Shadow Effect" where the anticipation of a future weaker contender makes strong players more aggressive (see Brown and Minor (2011)).

If he does not borrow, his expected payoff will be
\[
\pi^N_B = p_2(G^1, M_2) - M_2 + \delta\bar{\pi}. \quad (7.5)
\]

Summarizing, the dynamic best reply for the poor agent is:
\[
RD_2(G^1) = \begin{cases} 
RO_2(G^1) & \text{if } RO_2(G^1) \leq M_2 \\
RD^B_2(G^1) & \text{if } RO_2(G^1) > M_2, \text{ and } \pi^B > \pi^N_B \\
M_2 & \text{if } RO_2(G^1) > M_2, \text{ and } \pi^B \leq \pi^N_B
\end{cases}. \quad (7.6)
\]

7.2. Proof of Proposition 1

For each \(M_2 \in (0, \bar{G})\) let \(G^1 = M_2, \hat{G}^1 = RO_1(M_2)\), and \(\hat{G}^1 = RD^B_2(\hat{G}^1)\). An equilibrium without borrowing will exist if and only if
\[
\pi^N_B(\hat{G}^1, M_2) \geq \pi^B(\hat{G}^1, \hat{G}^1_2). \quad (7.7)
\]

Let \(F(M_2) = \pi^N_B(\hat{G}^1, M_2) - \pi^B(\hat{G}^1, \hat{G}^1_2)\). This function is continuous in \(M_2\).

Note that by strict concavity of expected payoffs
\[
\pi^B(\hat{G}^1, \hat{G}^1_2) > \pi^B(\hat{G}^1, RO_2(\hat{G}^1)). \quad (7.8)
\]

Let us see first that when \(M_2\) tends to zero, \(F(M_2)\) is negative. Since \(\hat{G}^1 = RO_1(M_2)\), by assumption (6.9), \(\lim_{M_2 \to 0} p_2(\hat{G}^1_1, M_2) = 0\), which implies that \(\lim_{M_2 \to 0} \pi^N_B = \delta\bar{\pi}\). Furthermore, since \(\lim_{M_2 \to 0} \hat{G}^1_1 = 0\), \(\lim_{M_2 \to 0} p_2(\hat{G}^1_1, RO_2(\hat{G}^1_1)) = 1\). Then, given that \(RO_2(\hat{G}^1_1) < V\),
\[
\lim_{M_2 \to 0} \pi^B(\hat{G}^1_1, RO_2(\hat{G}^1_1)) = (V + \delta\bar{\pi}) - RO_2(\hat{G}^1_1) > \delta\bar{\pi}. \quad (7.9)
\]
Thus, \(F(M_2)\) is negative.

Secondly, let us show that \(F(M_2)\) is positive when \(M_2\) tends to \(\bar{G}\). Recall that \(\bar{G}\) is the equilibrium effort for both players in the one shot game without any constraints considerations. Thus, \((\hat{G}^1_1, M_2)\) tends to \((\bar{G}, \bar{G})\) and therefore,
\[
\lim_{M_2 \to \bar{G}} \pi^N_B(\hat{G}^1_1, M_2) = \bar{\pi} + \delta\bar{\pi}. \quad (7.10)
\]
Since
\[ \tilde{\pi} > p(\tilde{G}, RD_G^B(\tilde{G}))V - RD_G^B(\tilde{G}), \]  
(7.11)
then
\[ \tilde{\pi} + \delta \tilde{\pi} > p(\tilde{G}, RD_G^B(\tilde{G}))V - RD_G^B(\tilde{G}) + \delta p(\tilde{G}, RD_G^B(\tilde{G}))\tilde{\pi}. \]  
(7.12)
Thus,
\[ \lim_{M_2 \to G} \pi^{NB}_2(G_1^1, M_2) > \lim_{M_2 \to G} \pi^B_2(G_1^1, \tilde{G}_2^1). \]  
(7.13)
Therefore, when \( M_2 \) tends to \( \tilde{G} \), \( F(M_2) \) is positive.

Summarizing,
\[ \lim_{M_2 \to 0} F(M_2) < 0, \text{ and } \lim_{M_2 \to \tilde{G}} F(M_2) > 0. \]  
(7.14)
By the intermediate value theorem, there exist \( M_2^{NB}(\delta) \) such that \( F(M_2^{NB}(\delta)) = 0. \)

Finally, let us see that \( F(M_2) \) is increasing in \( M_2 \). Denoting by \( F'(M_2) \) the derivative of \( F(\cdot) \) with respect to \( M_2 \), we have that
\[ F'(M_2) = \left( \frac{\partial \pi^{NB}_2(G_1^1, M_2)}{\partial G_1^1} \frac{\partial \hat{G}_1^1}{\partial M_2} + \frac{\partial \pi^{NB}_2(G_1^1, M_2)}{\partial G_2^1} \right) - \left( \frac{\partial \pi^B_2(G_1^1, \tilde{G}_2^1)}{\partial G_1^1} \frac{\partial \hat{G}_1^1}{\partial M_2} + \frac{\partial \pi^B_2(G_1^1, \tilde{G}_2^1)}{\partial G_2^1} \frac{\partial \hat{G}_2^1}{\partial M_2} \right). \]  
(7.15)
First, note that since \( M_2 < RO_2(G_1^1) \) then \( \partial \hat{G}_1^1/\partial M_2 > 0 \) and \( \partial \pi^{NB}_2(G_1^1, M_2)/\partial G_2^1 > 0 \). Also note that since \( \hat{G}_2^1 = RD_G^B(G_1^1) \), \( \partial \pi^B_2(G_1^1, \tilde{G}_2^1)/\partial G_2^1 = 0 \). Thus, \( F'(M_2) \geq 0 \) if
\[ \frac{\partial \pi^{NB}_2(G_1^1, M_2)}{\partial G_1^1} - \frac{\partial \pi^B_2(G_1^1, \tilde{G}_2^1)}{\partial G_1^1} \geq 0. \]  
(7.16)
Note that
\[ \frac{\partial \pi^{NB}_2(G_1^1, M_2)}{\partial G_1^1} = \frac{\partial \pi^{NB}_2(G_1^1, M_2)}{\partial G_1^1} V, \text{ and } \]  
(7.17)
\[ \frac{\partial \pi^B_2(G_1^1, \tilde{G}_2^1)}{\partial G_1^1} = \frac{\partial \pi^B_2(G_1^1, \tilde{G}_2^1)}{\partial G_1^1} (V + \delta \tilde{\pi}). \]  
(7.18)
Given that \( \partial p_2(\hat{G}_1^1, M_2)/\partial G_1^1 = -\partial p_1(\hat{G}_1^1, M_2)/\partial G_1^1 \),
\[ \frac{\partial p_2(\hat{G}_1^1, M_2)}{\partial G_1^1} = -\frac{\partial p_1(\hat{G}_1^1, M_2)}{\partial G_1^1} = -\frac{1}{V}. \]  
(7.19)
Also, \( \partial p_2(\hat{G}_1^1, \hat{G}_2^1)/\partial G_1^1 = -\partial p_1(\hat{G}_1^1, \hat{G}_2^1)/\partial G_1^1 \), \( G_1^1 = RO_1(M_2) < \tilde{G} \), and since \( \hat{G}_2^1 = RD_G^B(G_1^1) \), \( \hat{G}_1^1 < RD_G^B(\hat{G}_2^1) \), then
\[ -\frac{\partial p_1(\hat{G}_1^1, \hat{G}_2^1)}{\partial G_1^1} \leq -\frac{\partial p_1(RD_G^B(\hat{G}_2^1), \hat{G}_2^1)}{\partial G_1^1} = -\frac{1}{V + \delta \tilde{\pi}}. \]  
(7.20)
Thus, (7.16) holds and consequently \( F'(M_2) \geq 0 \).

Therefore, \( M^NB_2(\delta) \) is unique in \((0, G)\) and for all \( M_2 \geq M^NB_2(\delta) \) we have an equilibrium without borrowing. ■

7.3. Proof of Proposition 2

Let \( \tilde{G}^1_1 = RD^B_1(\tilde{G}^1_2) \) and \( \tilde{G}^1_2 = RD^B_2(\tilde{G}^1_1) \). Under our assumptions on \( p_i(.) \), \((\tilde{G}^1_1, \tilde{G}^1_2)\) exist. Recall that when agent 2 borrows, the game can be reinterpreted as one where agents have different valuations of the prize. Concretely, \( V_1 = (1 + \delta) V - \delta \bar{p}, \ V_2 = V + \delta \bar{p} \) and \( V_1 > V_2 \). Thus, \((\tilde{G}^1_1, \tilde{G}^1_2)\) is such that \( \tilde{G}^1_1 > \tilde{G}^1_2 \). That is, strategies are strategic complements for agent 1 and strategic substitutes for agent 2. Since \( M_2 < \tilde{G}^1_1 \), \( RD^B_1(M_2) < RD^B_1(\tilde{G}^1_2) = \tilde{G}^1_1 \).

First, we show that there exists \( M^B_2(\delta) \in (0, RO_2(\tilde{G}^1_1)) \) such that for all \( M_2 < M^B_2(\delta) \) an equilibrium with borrowing exists.

Given \( M_2 \in (0, RO_2(\tilde{G}^1_1)) \), an equilibrium with borrowing will exist if and only if
\[
\pi^B_2(\tilde{G}^1_1, \tilde{G}^1_2) \geq \pi^NB_2(\tilde{G}^1_1, M_2). \tag{7.21}
\]

Let \( H(M_2) = \pi^B_2(\tilde{G}^1_1, \tilde{G}^1_2) - \pi^NB_2(\tilde{G}^1_1, M_2) \). This function is continuous in \( M_2 \). Given that \( \pi^NB_2(\tilde{G}^1_1, M_2) = p_2(\tilde{G}^1_1, M_2)V - M_2 + \delta \bar{p} \), and \( RD^B_1(M_2) < \tilde{G}^1_1 \), \( \pi^NB_2(\tilde{G}^1_1, M_2) < \pi^NB_2(RD^B_1(M_2), M_2) \). By assumption (6.9), \( \lim_{M_2 \to 0} \pi^NB_2(RD^B_1(M_2), M_2) = \delta \bar{p} \). Thus, \( \lim_{M_2 \to 0} \pi^B_2(\tilde{G}^1_1, M_2) \leq \delta \bar{p} \). Since \( \delta \) is such that \( \pi^B_2(\tilde{G}^1_1, \tilde{G}^1_2) > \delta \bar{p} \), \( \lim_{M_2 \to 0} H(M_2) > 0 \). Let us show that \( \lim_{M_2 \to RO_2(\tilde{G}^1_1)} H(M_2) \) is negative. Since \( \pi^B_2(\tilde{G}^1_1, \tilde{G}^1_2) = p_2(\tilde{G}^1_1, \tilde{G}^1_2)V - \tilde{G}^1_2 + p_2(\tilde{G}^1_1, \tilde{G}^1_2)\delta \bar{p} \), we get that \( \pi^B_2(\tilde{G}^1_1, \tilde{G}^1_2) < p_2(\tilde{G}^1_1, \tilde{G}^1_2)V - \tilde{G}^1_2 + \delta \bar{p} \). But since \( p_2(\tilde{G}^1_1, \tilde{G}^1_2)V - \tilde{G}^1_2 + \delta \bar{p} < p_2(\tilde{G}^1_1, RO_2(\tilde{G}^1_1))V - RO_2(\tilde{G}^1_1) + \delta \bar{p} = \lim_{M_2 \to RO_2(\tilde{G}^1_1)} \pi^NB_2(\tilde{G}^1_1, M_2) \), we conclude that \( \lim_{M_2 \to RO_2(\tilde{G}^1_1)} H(M_2) < 0 \). Summarizing,
\[
\lim_{M_2 \to 0} H(M_2) > 0, \quad \lim_{M_2 \to RO_2(\tilde{G}^1_1)} H(M_2) < 0. \tag{7.22}
\]

By the intermediate value theorem, there exist \( M^B_2(\delta) \) such that \( H(M^B_2(\delta)) = 0 \). Finally, note that for all \( M_2 \in (0, RO_2(\tilde{G}^1_1)) \),
\[
H'(M_2) = -\frac{\partial \pi^NB_2(\tilde{G}^1_1, M_2)}{\partial G^1_2} < 0. \tag{7.23}
\]

Thus, \( H(M_2) \) is decreasing in \( M_2 \). Therefore, \( M^B_2(\delta) \) is unique and for all \( M_2 < M^B_2(\delta) \) we have an equilibrium with borrowing.
To complete this part, notice that if \( RO_2(\tilde{G}_1) < G \), then we already have the result. If \( RO_2(\tilde{G}_1) > G \), and \( G < M_2^B(\delta) < RO_2(\tilde{G}_1) \), then, since for all \( M_2 < M_2^B(\delta) \) we have an equilibrium with borrowing, for all \( M_2 \leq \tilde{G} \) we have an equilibrium with borrowing.

Finally, notice that, by non borrowing, agent 2 can always guarantee himself at least \( \delta \pi \) (ex- pending nothing in the first period will give him this payoff). Thus, if by borrowing he always get less than that, an equilibrium with borrowing will not exist.  

8. Appendix 3. Non existence of equilibrium in the Poor Man-Rich Man scenario

Assume that the CSF is given by (2.5). Let \( V = 100, \delta = 0.5, \) and \( M_2 = 10 \). In order to simplify notation we denote the expenses of agents 1 and 2 in the first period as \( G_1 \) and \( G_2 \). The best reply function for agent 1 is:

\[
G_1 = \begin{cases} 
\sqrt[3]{100G_2} - G_2 & \text{if } G_2 \leq M_2 \\
\sqrt[3]{137.5G_2} - G_2 & \text{if } G_2 > M_2 
\end{cases}
\]

For the poor agent \( G_2 = \sqrt[3]{100G_1} - G_1 \) if \( G_1 \) is such that \( \sqrt[3]{100G_1} - G_1 \leq M_2 \). That is, if \( G_1 \in [0, 1.27] \cup [78.73, \infty) \). For any other \( G_1 \), the poor agent has two options: either he borrows or he does not. If he borrows his expected payoff is given by:

\[
\pi_2^B = p_2^1V - \sqrt{V(1 + \frac{1}{4}\delta)G_1 + G_1 + \delta(p_2^1V^4)}.
\]

If he does not borrow, his expected payoff is given by

\[
\pi_2^{NB} = \frac{M_2}{M_2 + G_1}V - M_2 + \frac{\delta V}{4}.
\]

So, he will borrow if \( G_1 \) is such that \( \pi_2^B > \pi_2^{NB} \) which after some calculations amounts to

\[
\frac{\sqrt[3]{112.5G_1} - G_1}{\sqrt[3]{112.5G_1}} 112.5 - \sqrt[3]{112.5G_1} + G_1 > \frac{10}{10 + G_1} 100 + 2.5.
\]

That is, he will borrow if \( G_1 \in [3.82, 26.18] \). Thus, the best reply of the poor agent is:

\[
G_2 = \begin{cases} 
\sqrt[3]{100G_1} - G_1 & \text{if } G_1 \in [0, 1.27] \cup [78.73, \infty) \\
M_2 = 10 & \text{if } G_1 \in [1, 27, 3.82] \cup [26.18, 78.73] \\
\sqrt[3]{112.5G_1} - G_1 & \text{if } G_1 \in (3.82, 26.18)
\end{cases}
\]

In what follows we represent by a dashed line the best reply of agent 1 and the best reply of the poor agent by a solid line. Clearly, there is no a Nash equilibrium in pure strategies.
9. Appendix 4. Proofs of Propositions 3 and 4

Here we gather the proofs corresponding to the results obtained in the poor man-poor man scenario.

9.1. Proof of Proposition 3

Let $M < \bar{G}$. In the poor man-poor man case where both agents have identical endowments, if an equilibrium without borrowing exists, it should be such that $G_1^1 = G_2^1 = M$. An equilibrium without borrowing will exist if and only if

$$\pi_i^N(M, M) \geq \pi_i^B(\hat{G}_i^1, M), \quad (9.1)$$

where $\hat{G}_i^1 = RD_i^B(M)$. That is, even the best possible deviation when agent $i$ borrows and face bankruptcy, is not profitable. Without loss of generality, we do the analysis for agent 1. Let

$$F(M) = \pi_1^N(M, M) - \pi_1^B(\hat{G}_1^1, M).$$

This function is continuous in $M$. Note first that, since $\hat{G}_1^1 = RD_1^B(M),

$$\pi_1^B(\hat{G}_1^1, M) > \pi_1^B(\bar{G}, M). \quad (9.2)$$

Thus, $F(M) < \pi_1^N(M, M) - \pi_1^B(\bar{G}, M)$. Let us see first that when $M$ tends to zero, $F(M)$ is negative. Given that $\pi_1^N(M, M) = 1/2V - M + \delta\bar{M}, \lim_{M \to 0} \pi_1^N(M, M) = 1/2V + \delta\bar{M}$. Furthermore, $\lim_{M \to 0} p_1(\bar{G}, M) = 1$, which implies that $\lim_{M \to 0} \pi_1^B(\bar{G}, M) = V - \bar{G} + \delta\bar{M}.$ Therefore,
\[ \lim_{M \to 0} F(M) < -1/2V + \bar{G}. \] Given that \( G < 1/2V \), when \( M \) tends to zero, \( F(M) \) is negative.

Let us show that when \( M \) tends to \( \bar{G} \), \( F(M) \) is positive. Note first that \( \lim_{M \to \bar{G}} \pi_i^{NB}(M, M) = \bar{\pi} + \delta \bar{\pi}, \) and \( \lim_{M \to \bar{G}} \pi_i^B(G_1, M) = \pi_i^B(\lim_{M \to \bar{G}} B(G_1), \bar{G}) = p_1(\lim_{M \to \bar{G}} B(G_1), \bar{G})V - \bar{G}_1^B(\bar{G}) + \delta p_1(B(G_1), \bar{G})\bar{\pi}. \)

Given that
\[ \bar{\pi} \geq p_1(\lim_{M \to \bar{G}} B(G_1), \bar{G})V - \bar{G}_1^B(\bar{G}), \quad (9.3) \]
and \( p_1(\lim_{M \to \bar{G}} B(G_1), \bar{G}) \leq 1, \pi_i^B(\lim_{M \to \bar{G}} B(G_1), \bar{G}) \leq \bar{\pi} + \delta \bar{\pi}, \) thus, \( \lim_{M \to 0} F(M) > 0. \)

Summarizing,
\[ \lim_{M \to 0} F(M) < 0, \quad \text{and} \quad \lim_{M \to \bar{G}} F(M) > 0. \quad (9.4) \]

By the intermediate value theorem, there exist \( M^{NB}(\delta) \) such that \( F(M^{NB}(\delta)) = 0. \)

Finally, let us see that \( F(M) \) is increasing in \( M \). Denoting by \( F'(M) \) the derivative of \( F(\cdot) \) with respect to \( M \), we have that
\[ F'(M) = \left( \frac{\partial \pi_i^{NB}(M, M)}{\partial G_1^1} + \frac{\partial \pi_i^{NB}(M, M)}{\partial G_2^1} \right) - \left( \frac{\partial \pi_i^B(G_1^1, M)}{\partial G_1^1} \frac{\partial G_1^1}{\partial M} + \frac{\partial \pi_i^B(G_1^1, M)}{\partial G_2^1} \right). \quad (9.5) \]

Since \( \dot{G}_1^1 = \lim_{M \to \bar{G}} B(G_1^1), \partial \pi_i^B(G_1^1, M)/\partial G_1^1 = 0. \) Thus, \( F'(M_2) \geq 0 \) if and only if
\[ \frac{\partial \pi_i^{NB}(M, M)}{\partial G_1^1} + \frac{\partial \pi_i^{NB}(M, M)}{\partial G_2^1} - \frac{\partial \pi_i^B(G_1^1, M)}{\partial G_1^1} \geq 0. \quad (9.6) \]

Note first that
\[ \frac{\partial \pi_i^{NB}(M, M)}{\partial G_1^1} + \frac{\partial \pi_i^{NB}(M, M)}{\partial G_2^1} = -1, \]
because \( \partial \pi_i^{NB}(M, M)/\partial G_1^1 = (\partial p_1(M, M)/\partial G_1^1)V - 1, \partial \pi_i^{NB}(M, M)/\partial G_2^1 = (\partial p_1(M, M)/\partial G_2^1)V, \)
\[ \partial p_1(M, M)/\partial G_1^1)V = (-\partial p_2(M, M)/\partial G_2^1)V, \] and by symmetry, \( \partial p_2(M, M)/\partial G_2^1 = \partial p_1(M, M)/\partial G_1^1. \)

Finally, let us see that
\[ -1 - \frac{\partial \pi_i^B(G_1^1, M)}{\partial G_2^1} \geq 0. \quad (9.7) \]

Given that \( \partial \pi_i^B(G_1^1, M)/\partial G_2^1 = (\partial p_1(G_1^1, M)/\partial G_2^1)(V + \delta \bar{\pi}) = (-\partial p_2(G_1^1, M)/\partial G_2^1)(V + \delta \bar{\pi}). \)

Let \((G^*, \bar{G}^*)\) be the Nash equilibrium of the game with payoffs \( \pi_i(G_1, G_2) = p_i(G_1, G_2)(V + \delta \bar{\pi}) - G_i \)
without constraints considerations. Since \( V + \delta \bar{\pi} > V, G^* > \bar{G}. \) Furthermore, since \( \dot{G}_1^1 = \lim_{M \to \bar{G}} B(M), \)
and \( M < \bar{G} < G^*, \dot{G}_1^1 < G^*. \) Thus, we are in the strategic complements zone, and there, the best reply of agent 2 to \( \dot{G}_1^1 \) in this game is greater than \( \dot{G}_1^1 \) and therefore, greater than \( M. \) Thus,
\[ \partial p_2(\dot{G}_1^1, M)/\partial G_2^1)(V + \delta \bar{\pi}) - 1 \geq 0, \]

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Consider now the following payo\$ function:

Thus, \( F'(M_2) \geq 0 \). Therefore, \( M^{NB}(\delta) \) is unique in \((0, \bar{G})\) and for all \( M \geq M^{NB}(\delta) \) we have an equilibrium without borrowing. ■

9.2. Proof of Proposition 4

If both agents borrow, who ever wins the prize in the first period gets the entire prize in the second period without effort. Thus, the payoff will be

\[
\pi_i(G_1^1, G_2^1) = p_i(G_1^1, G_2^1)(V + \delta V) - G_i^1.
\]  

(9.8)

Let \((\bar{G}, \bar{G})\) be the Nash equilibrium in the static game defined by the payoffs in (9.8). (Under our assumptions on \( p_i(.) \), \((\bar{G}, \bar{G})\) exist and it is unique). Let us see that there exists \( M^B(\delta) \in (0, \bar{G}) \) such that \((\bar{G}, \bar{G})\) is an equilibrium with borrowing of our two period game if and only if \( M \leq M^B(\delta) \).

An equilibrium with borrowing will exist if and only if

\[
\pi_i^B(\bar{G}, \bar{G}) \geq \pi_i^{NB}(M, \bar{G}) \quad \text{for } i \in \{1, 2\}.
\]  

(9.9)

Without loss of generality, we do the analysis for agent 1.

Let \( H(M) = \pi_1^B(\bar{G}, \bar{G}) - \pi_1^{NB}(M, \bar{G}) \). This function is continuous in \( M \). Let us see first that when \( M \) tends to zero, \( H(M) \) is positive. Note first that \( \lim_{M \to 0} p_1(M, \bar{G}) = 0 \). Thus, given that \( \pi_1^{NB}(M, \bar{G}) = p_1(M, \bar{G})V - M + \delta(p_1(M, \bar{G})V + p_2(M, \bar{G})\bar{\pi}) \), \( \lim_{M \to 0} \pi_1^{NB}(M, \bar{G}) = \delta \bar{\pi} \). Since \( \delta \) is such that \( \pi_1(\bar{G}, \bar{G}) \geq \delta \bar{\pi}, \) (9.9) holds.

Consider now the following payoff function:

\[
\pi_1^{NB}(G_1, G_2) = p_1(G_1, G_2)V - G_1 + \delta(p_1(G_1, G_2)V + p_2(G_1, G_2)\bar{\pi}),
\]  

(9.10)

and let \( G_i^* \) the best reply to \( \bar{G} \) according to the above payoff. Let us see that when \( M \) tends to \( G_i^* \), \( H(M) \) is negative. By the definition of \( G_i^* \), \( \pi_1^{NB}(G_i^*, \bar{G}) \geq \pi_1^{NB}(\bar{G}, \bar{G}) = 1/2V - \bar{G} + \delta(1/2V + 1/2\bar{\pi}) \). Thus,

\[
\pi_1^{NB}(G_i^*, \bar{G}) > 1/2V - \bar{G} + \delta(1/2V) = \pi_1^B(\bar{G}, \bar{G}).
\]  

(9.11)

Therefore, \( \lim_{M \to G_i^*} H(M) < 0 \). By the intermediate value theorem, there exist \( M^B(\delta) \in (0, G_i^*) \)

such that \( H(M^B(\delta)) = 0 \). Finally, note that for all \( M \in (0, G_i^*) \),

\[
H'(M) = -\frac{\partial \pi_1^{NB}(M, \bar{G})}{\partial G_i^1} < 0.
\]  

(9.12)
Thus, \(H(M)\) is decreasing in \(M\). Therefore, \(M^B(\delta)\) is unique and for all \(M \leq M^B(\delta)\), \(H(M) \geq 0\).

To complete this part note that if \(M^B(\delta) < G\), we already have the result (that will be the case if for example \(G_1^* < \tilde{G}\)). If \(M^B(\delta) > \tilde{G}\), then for all \(M < \tilde{G}\) we have an equilibrium with borrowing.

\[\square\]

10. Appendix 5. Coexistence of borrowing and non borrowing equilibria

In order to gain intuition on the situation where both kind of equilibria coexist, we present the following example.

**Example 5.** Suppose that \(V = 80\), \(\delta = 1\), \(M_1 = M_2 = 10\) and \(q = 8\). Both agents are constrained. In order to simplify notation we denote the expenses of agents 1 and 2 in the first period as \(G_1\) and \(G_2\). Since both agents are identical, their best reply functions are identical too.

\[
G_i = \begin{cases} 
\sqrt[4]{VG_j - G_j} & \text{if } G_j \leq M_i \text{ and } \sqrt[4]{VG_j - G_j} < M_i \\
M_i & \text{if } G_j \leq M_i \text{ and } \sqrt[4]{VG_j - G_j} > M_i, \text{ and } \pi_i^{NB} > \pi_i^B \\
\sqrt[4]{V(1 + \frac{\delta}{4})G_2 - G_2} & \text{if } G_j \leq M_i \text{ and } \sqrt[4]{VG_j - G_j} > M_i \text{ and } \pi_i^{NB} < \pi_i^B \\
M_1 & \text{if } G_j > M_i \text{ and } \pi_i^{NB} > \pi_i^B \\
\sqrt[4]{V(1 + \delta)G_2 - G_2} & \text{if } G_j > M_i \text{ and } \pi_i^{NB} < \pi_i^B 
\end{cases}
\]

where

\[
\pi_i^{NB} = \frac{M_i}{M_i + G_j}V - M_i + \frac{V}{4}, \quad (10.1)
\]

\[
\pi_i^B = \frac{\sqrt[4]{V(1 + \frac{\delta}{4})G_j - G_j}}{\sqrt[4]{V(1 + \frac{\delta}{4})G_j}}V(1 + \delta) - \sqrt[4]{V(1 + \frac{\delta}{4})G_j + G_j}, \quad (10.2)
\]

\[
\bar{\pi}_i^{NB} = \frac{M_i}{M_i + G_j}V(1 + \delta) - M_i, \quad (10.3)
\]

\[
\bar{\pi}_i^B = \frac{\sqrt[4]{V(1 + \delta)G_j - G_j}}{\sqrt[4]{V(1 + \delta)G_j}}V(1 + \delta) - \sqrt[4]{V(1 + \delta)G_j + G_j}. \quad (10.4)
\]

In this example, for any \(G_j \in [0, 10]\), \(\pi_i^{NB} > \pi_i^B\), and for any \(G_j \in (10, 80]\) \(\bar{\pi}_i^{NB} < \bar{\pi}_i^B\). The best reply functions of both agents are plotted in the following figure.
So we have an equilibrium without borrowing when both agents spend 10 and another equilibrium with borrowing where both agents spend 40. The first equilibrium is somehow not very robust because if, say, player 1 chooses $10 + \varepsilon$ ($\varepsilon > 0$ but very small) the best reply of player 2 is far away from 10. But this equilibrium is robust if endowments vary a little.

References


