Universidad Carlos III de Madrid
GAME THEORY
Problem set on static games

1. Two car companies are planning to launch at the same time a car to the market. Each of them is considering whether it should offer credit to the buyers in order to reach a larger share of customers. However; offering credit would imply incurring some costs. Both companies prefer not to offer credit but they are afraid that the other one will do offer and will therefore attract more clients. Suppose that the expected benefits for both companies are the following: If none of them offers credit, they get 800 million each. If both offer a credit, each gets 600 million, and if one offers credit and the other one does not; the first one will earn 900 million while the other will obtain 400.

(a) Represent the game in the normal form.
(b) Which game seen in class is similar to this one?

2. Freedonia is a country that needs financial aid, and that can only obtain it from the Pangean Union (PU), an economic union of countries where it belongs. As a condition for the aid, the PU requires that Freedonia makes some deep fiscal reforms. The government of Freedonia makes a counter proposal with a different package of conditions (basically, minor reforms). Each side beliefs that its proposal is the best, and that the other one will imply a disaster. They can hold their position or make some concessions to the other part. If they both make concessions, negotiations will end up with a result that is unsatisfactory for either of them, but still better that if no one makes concessions and hold their position, the worst possible outcome for both of them. Of course, each one would prefer to hold while the other party is the only one making concessions.

(a) Represent the game in the normal form.
(b) Which game seen in class is similar to this one?

3. The popular game *Rock-paper-scissors* is played between two players. Each player has to choose one of the three objects, and they win according to these rules: rock beats scissors, scissors beats paper and paper beats rock. In case they choose the same object, they are tied. Write the normal form of the game.

4. Guillermo and Miguel share an apartment where each one of them has his own room. They want to decorate their apartment. Each one has two paintings and must decide how many to place in his own room and how many to place in the common living room. Suppose that the decision is made privately and that once the paintings are in their place, they cannot be removed. Let x_G and x_M be the number of paintings that Guillermo and Miguel, respectively, decide to place in their own room (thus, $x_S = 4 - x_G - x_M$ is the number of paintings in the living room). Guillermo's utility function is $u_G(x_G, x_S) = x_G(1.5 + x_S)$ and Miguel's is $u_M(x_M, x_S) = x_M(1.5 + x_S)$. Then, for instance, if Miguel places one painting is his own room and Guillermo two in his ($x_M = 1, x_G = 2, x_S = 1$) they will get utilities $u_M = 2.5$ and $u_G = 5$.

(a) Which are the strategies of each one of the roommates?

(b) Represent the game. I.e., in the usual entry matrix describe the utilities of each player for each of the possible distributions of the paintings, depending on the chosen strategies.

5. Consider the prisoners' dilemma game seen in class.

(a) Change the payoffs to reflect the case in which one year in prison suffered by the partner causes as much pain as half a year suffered by oneself, but only if the partner did not confess.
(b) Repeat (a) for the case only Player 1 has the altruistic preferences described in (a), while Player 2 has the original preferences in the prisoners' dilemma.

6. Solve the following:

(a) Find the Nash equilibria in problems 4 and 5.
(b) In Problem 5, how much pain should cause a year in prison by the partner (always in case he did not confess) for the game to have only an equilibrium in which no one confesses.

7. Consider the following game in the normal form:

	L	R
T	4, 4	0, 2
M	2, 1	1, 0
B	a, b	c, 1

(a) For which values of a, b and c, the strategy profile (B, L) is the result of the iterated elimination of dominated strategies?
(a) For which values of a, b and c, the strategy profile (B, L) is the unique pure strategy Nash equilibrium?

8. The municipality of Madrid is organizing an operation called *Madrid Verde*. In a street of Chamartín, each family that has a house there will get two trees. Only two neighbors live on this street (only two houses). Each of the two neighbors must decide how many of these trees he will plant in his garden (in which case you cannot see the trees from the street) and how many he will plant in the entrance of his house (in which case you can see the trees from the street). The trees that can be seen from the street contribute to the revaluation/improvement of the neighborhood. Neighbor 2 values more than neighbor 1 the trees that can be seen from the street because he intends to sell his house soon. Suppose that the decision of each neighbor is private and that once they have planted the trees they cannot change their position. Let x_1 and x_2 denote the number of trees that Neighbor 1 and Neighbor 2, respectively, has decided to plant in his garden. Let xc denote the number of trees that can be seen from the street. The utility function of Neighbor 1 is given by $U_1(x_1, x_c) = x_1(1.5 + x_c)$ and that of Neighbor 2 by $U_2(x_2, x_c) = x_2(1.5 + ax_c)$, where $a > 1$.

(a) Represent the above game in normal form and find its Nash Equilibria in pure strategies.

(b) Do the equilibriums that you have found in question 1 above maximize the total social utility (the total social utility is given by $U_T(x_1, x_2) = U_1(x_1, x_c) + U_2(x_2, x_c)$? Justify your answer.

9. Two armies fight a war. Army 1 has 4 divisions, while Army 2 has only 3. The war is decided on two battlefields, and each army has to decide how many divisions to locate on each one, where the army with more divisions wins the battle. If the number of divisions is the same, then no one wins. Consider that the decisions are made simultaneously and that payoffs are calculated as the number of won battles minus the number of lost battles.

(a) Which are the strategies for each army? Indicate them as a pair of numbers, the first for the number of divisions on battlefield 1, and the second for divisions on battlefield 2.
(b) Write the normal form of the game.
(c) Iteratively eliminate weakly dominated strategies.
(d) Calculate the equilibrium in the game obtained in (c).

10. The treasurer and bookkeeper of a certain political party, Mr. Luis Barreras, has to launder one million euros, the bounty of some illicit activities. He can do it through a company with an opaque account in a tax haven or, alternatively, through different friends and family as front men. The first option costs one quarter of a million euros, while the second one costs half a million. The Tax Agency has resources to inspect just one of the two possibilities. If it inspects the option that Barreras did not use, he will keep his net benefits and the Tax Agency will not collect a single euro. If the Tax Agency inspects the option that Barreras has used, it will locate the money and will seize all that was left in Barreras' possession, and it will, in addition, impose a penalty of half a million euros.

(a) Represent the normal form of the game.
(b) Calculate the Nash equilibria.

11. Find the mixed strategy Nash equilibria for the game in Exercise 8.

12. Find the mixed strategy Nash equilibria for the game in Exercise 10.

13. Find the mixed strategy Nash equilibrium of the following normal form game:

	L	R
T	2, 1	0, 2
B	1, 2	3,0

14. Consider the game of *Rock-paper-scissors* in Exercise 3.

(a) Suppose that Player 2 plays *rock* and *paper* with probabilities ½ and ½. Which is Player 1's best reply?
(b) Show that $\left(\left(\frac{1}{3}[R], \frac{1}{3}[P], \frac{1}{3}[S]\right), \left(\frac{1}{3}[R], \frac{1}{3}[P], \frac{1}{3}[S]\right)\right)$ is a Nash equilibrium in mixed strategies.

15. Consider the game between Nadal and Federer seen in class, and add a third strategy for Federer, who now can return from the middle. Nadal, on the other hand, has only his original two strategies. If Federer uses his new strategy, he wins the point 30% of the times if Nadal serves towards the left and 35% if he serves towards the right.

(a) Represent the normal form of the new game.
(b) Show whether the new strategy *Middle* for Federer is dominated by some pure or mixed strategy.

16. Two firms (1 and 2) compete *a la* Cournot in a market with demand given by $p = 130 - q$. They have no fixed costs. Find the Nash equilibrium quantities, price and profits in the following cases:

(a) Variable costs are $c_i(q_i) = 10$.
(b) Variable costs are $c_1(q_1) = 10$, $c_2(q_2) = 25$.

17. Three firms (1, 2 and 3) compete *a la* Cournot in a market with demand given by $p = 130 - q$. All have the same variable costs $c_i(q_i) = 10$. Find the Nash equilibrium quantities, price and profits in the following cases:

(a) There are no fixed costs.
(b) Each firm has a fixed cost of 1000.

18. Four firms (1, 2, 3 and 4) compete *a la* Cournot in a market with demand given by $p = 130 - q$. All have the same variable costs $c_i(q_i) = 10$. Firms 3 and 4 are considering a merging, by which the would become Firm X. Show whether this merging is in the interest of firms 3 and 4 in the following cases:

(a) Firms have no fixed costs.
(b) Each firm has a fixed cost of 400. Note: in the case of a merging, the fixed cost is also 400 for the new Firm X.

19. Consider a market with two firms (1 and 2) that sell two slightly different products. The two demand functions are $q_1 = 1000 - 2p_1 + p_2$ and $q_2 = 1000 - 2p_2 + p_1$, respectively. Both firms have access to the same technology with zero variable costs zero and no fixed costs. Find the Nash equilibria of this game in the following cases:

(a) Firms compete *a la* Bertrand.
(b) Firms compete *a la* Cournot.

20. Two neighbors consider cleaning their street on a Sunday morning. Each neighbor has one hour that he can use to watch TV or to clean the street. Denote by c_i the time dedicated to cleaning the street and by Neighbor i, and $1 - c_i$ the time spent watching TV. Each neighbor considers that it is important that the street is clean and likes watching TV. Their utility functions are:

$$U_1(c_1, c_2) = 2\ln\left(1 + c_1 + \frac{c_2}{2}\right) + 1 - c_1$$
$$U_2(c_1, c_2) = 2\ln\left(1 + \frac{c_1}{2} + c_2\right) + 1 - c_2$$

where $2\ln\left(1+c_i+\frac{c_j}{2}\right)$ represents the utility of living in a clean street for Neighbor i and $1-c_i$ is the utility from watching TV for Neighbor i.

(a) Calculate the Nash equilibria of the game.
(b) Represent graphically the best response functions and the Nash equilibria.

Extra Exercises

21. Consider two firms, one of them operates in a market (Firm 1) and the other one (Firm 2) wants to enter in that market. Firm 1 is thinking about building a new production plant. Payoffs for the two firms are given by:

	Enter	Do not enter
Build	0, -1	2, 0
Do not build	2, 1	3, 0

Explain the above payoffs. Find the Nash equilibria of this game.

22. Carlos (C) and Pepe (P) want to divide 1000 euros between them. Both of them announce simultaneously which portion of the 1000 euros they would like to keep, s_i $(i = C, P)$ with $0 \leq s_i \leq 1000$. If $s_C + s_P \leq 1000$, each one receives what he has asked for. In the opposite case, both receive 0. Which are the pure strategy Nash equilibria of this game?

23. Six siblings have to decide who will take the family car for the weekend. They have decided to use the following procedure. Each of them will write simultaneously a number between 0 and 10. They will then compute the average number. The car will go to the sibling that has written the closest number to the average but smaller than the average. In the case of a tie, their father will choose who takes the car randomly. Find the Nash equilibria of this game. Provide an argument for your answers.

24. There are two ice-cream sellers on a beach. The prices and the quality of their ice creams are the same. The unique decision taken by each seller is his location on the beach. Consumers are uniformly distributed along the space and direct themselves to the closest seller. Sellers choose their location in order to maximize the number of buyers they capture. The two locations (of the two sellers) that are socially optimal are the ones that minimize the total distance covered by all the consumers.

(a) Explain why the sellers do not want to stay at these socially optimal locations (The solution is obtained by thinking that these locations do not constitute a Nash equilibrium). In which direction are they willing to move? Where do they locate themselves at the end?
(b) Would your answer be affected if buyers consumed less ice creams when the distance they must cover increases?
(c) Suppose now that instead of two ice-cream sellers there are three. Show that there is no pure strategy equilibrium.

25. Two players write down, simultaneously, a number between 0 and 1. Let s_i be the number written down by Player i. The payments for each player depend on the difference between both numbers according to the following function:

$$u_1(s_1, s_2) = u_2(s_1, s_2) = (s_1 - s_2)^2$$

(a) Find the best response of Player 2 to the following strategies: $s_1 = 1$, $s_1 = 0$, and $s_1 = \frac{1}{2}$.
(b) Represent graphically each player's best response function.
(c) Find the Nash equilibria in pure strategies.

26. Two friends share an apartment. They have to decide how to divide a total of 8 hours in cleaning the apartment $(s_i, i = 1,2)$ and watching television $((8 - s_i, i = 1,2)$. Both value a clean apartment and both like to watch television (especially when the apartment is clean). Suppose that the utility function of player i is the following:

$$u_i(s_1, s_2) = (s_1 + s_2) + (8 - s_i) + (8 - s_i)(s_1 + s_2)$$

where the first term represents the utility from living in a clean apartment (which is a result of the time the friends decide cleaning the apartment), the second term is the direct utility from watching television and the last term reflects that the utility from watching television increases when the apartment is clean.

(a) Find the Nash equilibria of the game.
(b) Do the Nash equilibria of the game found in the previous question maximize the joint welfare of the friends: $u_1(s_1, s_2) + u_2(s_1, s_2)$?

27. Two villages that belong to two different independent communities compete for a plan that will offer tax reductions that help industrial development. If both offer the same reductions, the tax rate decreases with no guarantees of any industrial development. In such a case, the villages would have preferred higher taxes. The idea is to attract firms even if it implies tax reductions. Represent this situation as a game with a numerical example. Explain what are the relevant strategic factors.

28. Pedro and Miguel are neighbors. From his terrace, Pedro cannot see his own garden but he has a wonderful view of Miguel’s garden. Pedro considers that maintaining Miguel’s garden is worth 2000 euros. Pedro also considers that keeping his own garden in a good state is worth only 500 euros. The preferences of Miguel are completely reciprocal. Given that both gardens can be seen form the public road, the mayor pays a subsidy of 500 euros to each house in a street in which all the gardens are maintained. Pedro and Miguel are the two only neighbors of their street. The cost to maintain each garden is 1000 euros. Represent the game faced by Pedro and Miguel.

29. The city hall has to decide whether to construct a high school or a nursery in a particular city zone. It does not have budget to carry out both projects (i.e. both the high school and the nursery). The person in charge of managing these subjects has spoken with two indispensable companies that can make any of these two projects: one construction company and one carpentry company. Due to the composition of population, the building of the high school would be greater than the one of nursery (it requires more construction), but that one will need a park of wood games (it requires more carpentry). In addition, each one of the companies is interested more in participating in a certain project that in the other (the one of construction in the high school, and the one of carpentry in the nursery), but both prefer signing the same contract to signing different contracts, since in this case the city hall would not carry out any project. The city hall asks them to present a project. As none of the companies has sufficient personnel available to process both projects, they must choose one of the projects another, without knowing which project will be chosen by the other company.

(a) Define a game in normal form whose payments reflect the expected profits of each company in each possible situation.
(b) Which game seen in class is closer to this one?

30. Consider the following game in the normal form:

	B_1	B_2	B_3
A_1	1, 1	0, 0	-1, 0
A_2	0, 0	0, 6	10, -1
A_3	2, 0	10, -1	-1, -1

(a) Which strategies survive to the iterated elimination of dominated strategies?
(b) Which are the Nash equilibria?

31. Two friends, Andrea and Bernardo, would like to share 101 Euros. They agree on the following procedure. Each friend writes (without knowing what the other writes) a number on a sheet of paper. They share the money according to the following rules:

i. If both numbers are even, Andrea obtains 51 Euros and Bernardo 50;
ii. If both numbers are odd, Andrea obtains 50 Euros and Bernardo 51; and
iii. In the remaining cases, both obtain 50 Euros, and one Euro is lost.

(a) Represent this game (Game A) in normal form.
(b) Find the pure strategy Nash equilibria.

Bernardo is specially interested in the egalitarian solution, but he does not like the one that would be obtained from the previous game. He proposes to give each player more options by introducing the option *Don't write a number* (denote this new game by Game B). In this case if one player writes a number and the other not, the player not writing a number obtains all the money and the other nothing. On the other hand, if both choose the strategy *Don't write a number*, each player obtains one euro and the rest of the money is lost.

(c) Find the Nash equilibria of the new game (Game B).

32. Consider the following game in the normal form:

	B_1	B_2	B_3
A_1	2, 0	1, 1	4, 2
A_2	3, 4	1, 2	2, 3
A_3	1, 3	0, 2	3, 0

(a) Which strategies survive to the iterated elimination of dominated strategies?
(b) Which are the Nash equilibria?

33. In a small island there are only two consumers and two importing cars firms. One firm imports American cars while the other imports European cars. Right now, both consumers are without a car, and since there is no public transport services in the island, each of them is going to buy a car, no matter the price. But they don't want to expend too much, so they both are going to buy from the firm with the lowest price. If both firms charge the same price, each one buys from a different firm.

When a firm sells a car, it has to pay the import cost, which is equal to 10 000 if the car comes from USA, or 8000 if from Europe. The firms cannot fix prices with decimals, since the smaller coin in the island is of one monetary unit. The firms mainly want to maximize profits, and if at the same profits, they prefer to sell as much as possible.

(a) For the case where the two firms choose their prices simultaneously, write the payoff functions for each of them.
(b) If the firm who sells American cars sets a price $p_A = 10\,000$, what is the best response for the other firm? And if $p_A = 5000$?

Note: If a firm is indifferent among many different prices and one of them is equal to the import cost, we assume that it sets a price equal to its import cost.

(c) If the firm who sells European cars sets a price $p_E = 15\,000$, what is the best response for the other firm? And if $p_E = 1000$?
(d) Find the reaction functions for each firm, for all possible prices from its rival. Find the only Nash equilibrium in pure strategies. How many cars each firm sells, and at which price?

Suppose now that the government of the island is worried about the prices of the cars, and decides to subsidize one (and only one) of the firms, paying 1000 of its imports costs for each car it sells.

(e) If the government's objective is that the consumers buy the cars at the lowest possible price, to which firm should the government give the subsidy?
(f) Who is going to sell now? At which price? And how much the government will expend in subsidies?