Universidad Carlos III de Madrid
GAME THEORY
List of exercises for repeated and Bayesian games

1. Consider the chicken game in the course web page: Notes and links > Finitely Repeated Games > Slide 14. Suppose that the game is repeated three times. Find a trigger strategy that sustains (S, S) in the first two repetitions and that is also a subgame perfect Nash equilibrium. Note: make sure that the trigger strategy precisely defines what to play in each possible subgame.

2. Consider the following game in the normal form:

		Player 2		
		C	N	P
	C	6, 6	0, 9	0, 0
Player 1	N	9, 0	3, 3	0, 0
	P	0, 0	0, 0	1, 1

(a) Find the pure strategy Nash equilibria.

Suppose now that the game from part (a) is played twice. After the first time, the players observe what happened and play again, so they can make their strategies for the second round contingent to what occurred in the first round. The final payoffs are the sum of the payoffs from each round.

(b) Is there a subgame perfect Nash equilibrium where the players play (C, C) in the first round?

Suppose now that the game is repeated three times.

(c) Is there a subgame perfect Nash equilibrium where the players play (C, C) in the first round?
(d) Is there any where players play (C, C) in the first two rounds?
(e) Answer (c) and (d) if the game is repeated four times.

3. In the telecommunications market of a country there are two Firms that face a market demand given by

$$P(q_a, q_b) = 160 - q_a - q_b.$$

The two firms have the same costs of production $C(q) = 40q$.

(a) Find the Nash equilibria in this game when the two firms simultaneously select their quantities of production just one time.
(b) Find a discount rate and a subgame perfect Nash equilibrium such that the firms collude (they share equally the maximum possible joint benefits given the demand) if the game is repeated infinitely.
(c) Find a strategy such that in each period, the firms produce (q_a, q_b) with $q_a + q_b = Q^M$, where Q^M is the monopolist quantity and $q_a > q_b$, and that the strategy is a subgame perfect Nash equilibrium for some discount factor.

4. Consider the prisoner's dilemma described below:

	C	D
C	1, 1	5, 0
D	0, 5	4, 4

In the repeated game (finitely or infinitely many times) the strategy *tit for tat* (*toma y daca*) is defined like this:

- At $t = 1$ play (C, C).
- At $t > 1$ Player i plays what Player j played at $t - 1$ $(i, j = 1,2,\ i \neq j)$.

(a) Show that this strategy is not a Nash equilibrium in the finitely repeated game. Note: start by considering that the game is repeated twice.

Suppose now that the game is repeated infinitely many times. Note: in the questions below, just show the payoffs day by day. No need to calculate the discounted sum of payoff.

(b) What are players' payoffs if they follow *tit for tat*?
(c) What are the players' payoffs if Player 1 deviates in the first period?
(d) If Player 1 deviates in the first two periods?
(e) In the first three?
(f) In the first four?
(g) In the first five?
(h) In the first and the third?

5. There are two suspects who face the problem of the prisoner's dilemma, with the added complication that neither suspect knows if the other is a man of honor. Say it is known with certainty that Suspect 1 is not a man of honor, but it is not clear whether or not Suspect 2 is also. If Suspect 2 is not a man of honor, the payoffs have their usual form in this game:

		Suspect 2	
		Confess	*Not to Confess*
Suspect 1	*Confess*	1, 1	15, 0
	Not to Confess	0, 15	10, 10

On the other hand, if Suspect 2 is a man on honor, then he prefers to spend years in jail before he would rat on his colleague. More over, even Suspect 1 would feel bad betraying someone so honorable. For these reasons, if Suspect 2 is a man of honor the payoffs are:

		Suspect 2	
		Confess	*Not to Confess*
Suspect 1	*Confess*	1, 1	5, 20
	Not to Confess	0, 15	10, 30

Denote the probability that Suspect 2 is a man of honor by p. Consider $p \in (0,1)$.

(a) Identify the strategies that are dominant for Suspect 2 in this game.
(b) Identify the Bayesian-Nash equilibria in this game for each value of p.

6. A forward and a goalkeeper face each other in a penalty shootout. The forward is a very skilled shooter and, if he deceives the goalkeeper, he always scores a goal. The goalkeeper can either be very good or average with equal probabilities. If he is very good, he will save every shot unless the forward deceives him. If he is average, he will only save half of the shots unless he is deceived. More specifically, the forward can shoot to the left (L) or to the right (R). The goalkeeper can also dive to the left (L) or to the right (R). Deceiving the goalkeeper means, for example, the forward shooting to the left and the goalkeeper diving to the right, or vice versa. If the forward scores, it means a payoff of 1 for the forward and -1 for the goalkeeper. If he does not, payoffs are reversed.

(a) Show the Bayesian game, with all its elements defined (indicate the payoffs in matrix form).
(b) Show that the game does not have any Bayesian Nash equilibrium in pure strategies.
(c) Show that $\left(\frac{1}{2}[L] + \frac{1}{2}[R];\ \left(\frac{1}{2}[L] + \frac{1}{2}[R], \frac{1}{2}[L] + \frac{1}{2}[R]\right)\right)$ is a Bayesian-Nash equilibrium in mixed strategies. (The first component of the equilibrium is the forward's strategy and the second is the goalkeeper's strategy, with one action per type.)

7. Two friends share an apartment. They have to decide how to divide a total of 6 hours in cleaning the apartment, s_i and watching television, $6 - s_i$ $(i = 1,2)$. Both value a clean apartment and both like to watch television (especially when the apartment is clean). The utility function of Player 1 the following:

$$u_1(s_1, s_2) = (s_1 + s_2) + (6 - s_1) + (6 - s_1)(s_1 + s_2).$$

Player 2' utility is

$$u_2(s_1, s_2) = a(s_1 + s_2) + (6 - s_2) + (6 - s_2)(s_1 + s_2).$$

The first term represents the utility from living in a clean apartment, the second term is the direct utility from watching television and the last term reflects that the utility from watching television increases when the apartment is clean. The parameter a in Player 2's utility may take the values 1 and 4 with probabilities 1/3 and 2/3, respectively. Player 2 knows the value of a, but Player 1 does not, although she knows the probability distribution over the possible values.

(a) Identify all the elements of the above description to define a Bayesian game.
(b) Find the Bayesian Nash equilibria of the game.

8. Consider a Cournot duopoly in a market with inverse demand function given by $p(q) = 10 - q$, where $q = q_1 + q_2$ is the market total quantity. Firm 1's costs are $c_1(q_1) = 8q_1$ with probability 1/2, and $c_1(q_1) = 0$ with probability 1/2. Firm 2's costs are $c_2(q_2) = (q_2)^2$. Firm 1 knows its costs, but not Firm 2, who knows Firm 1's possible types and probabilities. All this is common knowledge.

(a) Represent the situation as a Bayesian game. I.e., show the set of players, the set of types, beliefs, and utilities.
(b) Calculate the Bayesian-Nash equilibrium of the game. Calculate also the equilibrium profits.
(c) Consider Firm 1's low-cost type ($c_1(q_1) = 0$), does it gain if it can inform Firm 2 of its type in a credible way (providing proof)? What about the high cost type? ¿How does this possibility affect the equilibrium?
(d) Assume that Firm 1 only has its word (no proof) to convince Firm 2 of its type. Does Firm 1 have any incentives to lie about its type?

Other Exercises

9. Consider the following game in the normal form:

	A	B	C
A	3, 3	x, 0	-1, 0
B	0, x	4, 4	-1, 0
C	0, 0	0, 0	1, 1

(a) Find the Nash equilibria in pure strategies for the different values of x.

Suppose now that the game is played twice. After playing the first time, the players observe what happened before playing the second time. The final payoffs are the sum of the payoffs from each round.

(b) Let $x = 5$. Is (B, B) a Nash equilibrium for the game when it is played just once? Is there a subgame perfect Nash equilibrium where (B, B) is played in the first round? If yes, write the strategies for both players for such equilibrium. If no, explain why not, using the definition of subgame perfect Nash equilibrium.
(c) Let $x = 7$. Is there a subgame perfect Nash equilibrium where (B, B) is played in the first round? If yes, write the strategies for both players for such equilibrium. If no, explain why no.

10. Given the following Normal form game:

	L	R
A	1, 1	8, 0
B	0, 5	3, 3

Find the smallest discount factor necessary to get average payoffs equal to (3,3) in a SPNE if the game is repeated an infinite number of times. Describe the strategies that allows us to sustain such an equilibrium

11. Consider a Cournot duopoly in a market with inverse demand function given by $p(q) = a - q$, where $q = q_1 + q_2$ is total market quantity. Both firms have cero costs. Demand is uncertain: it can be high ($a = 12$) or low ($a = 6$) with equal probabilities.

Information is asymmetric: Firm 1 knows whether the demand is high or low, but Firm 2 does not. All this is common knowledge.

(a) Represent this situation as a Bayesian game.
(b) Compute the Bayesian-Nash equilibrium and the equilibrium profits.

12. Carlos decided to have lunch at the restaurant Casa Pepe. The waiter of the restaurant has to decide whether to attend Carlos with a good service or with a bad service. Carlos, after observing the service received from the waiter, decides if he tips the waiter or not. The waiter likes to receive a tip, but he has a cost to provide a good service. Carlos, on the other hand, likes to receive a good service, although he does not like to tip waiters. Each one of them wants to maximize his own expected value. Suppose that the only possible tips are 2 or 0 euros. For Carlos, a good service is worth 6, while a bad service is worth nothing. For the waiter, a good service costs 1, while a bad service has no cost.

(a) Draw the extensive form of this game.
(b) Which are the pure strategies for Carlos?
(c) Is there a Nash equilibrium where Carlos pays the tip only if he receives a good service, and the waiter gives a good service? Explain.
(d) Represent this game in the normal form.
(e) Which of the following phrases is correct?
  i) For the waiter, to give a good service is a dominant strategy.
  ii) Never give a tip, no matter the quality of the service, is a dominant strategy for Carlos.
  iii) For Carlos, it is better to give a tip if the service is good, and do not if it is bad.
  iv) None of the above is correct.
(f) Find the pure strategy Nash equilibrium.
(g) Does the equilibrium in (f) result in a Pareto efficient allocation? If not, indicate a strategy profile that results in a Pareto superior allocation.

Suppose now that Carlos goes to this restaurant every week, and he is always served by the same waiter. Each one maximizes expected value, and no one discounts the future (i.e., one euro today is worth exactly the same as one euro in the future). In that case, they could reach the following verbal agreement: the waiter starts providing a good service, and will keep doing so in the future if he always receives a tip. If in some week Carlos does not give a tip, the waiter will never give a good service again. And Carlos will always give tips, as long as he receives a good service. If the waiter fails once, and give him a bad service, Carlos will stop tipping the waiter forever. If everybody knows that the waiter is leaving the restaurant the first day of next month.

(h) Will they follow the agreement in a subgame perfect Nash equilibrium.

13. Calculate the Bayesian Nash equilibria in pure strategies of the following static Bayesian game:

- Player 1 can choose between two actions *A* and *B*. Player 2 can choose between two actions, *I* and *D*.

- The payoffs depend on the players types. Player 1 has just one type and this is known by Player 2.
- Player 2 can be of either type x or y. Player 2 knows her own type but Player 1 does not know for certain the type of Player 2.
- Player 1 thinks that Player 2 is Type x with probability 2/3, and Type y with probability 1/3.
- The payoffs are those that are given in the game that is randomly determined.

Type x

	I	D
A	4, 1	3, 3
B	3, 6	2, 3

Type y

	I	D
A	3, 6	1, 3
B	1, 1	5, 3

14. The game of chicken will be familiar to those who have seen West Side Story or Rebel without a cause. One simplified version of this game is as follows. Two players drive their cars at each other at full speed. They can take one of two possible actions, swerve aside or continue straight on. The payoffs are as follows.

		James Dean	
		Continue	*Swerve*
Bad guy	*Continue*	-3, -3	2, 0
	Swerve	0, 2	1, 1

(a) Find the Nash equilibria of this game.

Now suppose that everyone knows that the preferences of Dean are those that are given above, but there exist serious doubts about the sanity of the Bad guy, to the point that some people believe that he will never swerve, not even if it might cost him his life. More formally, the payoffs are the same as above with probability p and are as follows with probability $1 - p$.

		James Dean	
		Continue	*Swerve*
Bad guy	*Continue*	2, -3	2, 0
	Swerve	0, 2	1, 1

(b) Find the pure strategies Bayesian equilibrium of this game.

15. Two bidders participate in an auction to buy a painting. Their values for the painting are either 20 or 100, both being equally likely. Each individual knows her valuation but does not the valuation of the other player. The acceptable bids are $\{10, 30, 50\}$. Consider only equilibria where players do not use weakly dominated strategies.

(a) Find the Bayesian-Nash equilibria of the first-price auction.
(b) Find the Bayesian-Nash equilibria of the second-price auction.
(c) Compare the auctioneer's revenue in each auction.

16. Consider the following Bertrand duopoly with differentiated products and asymmetric information. The demand for Firm i is $q_i(p_i, p_j) = 10 - p_i - b_i p_j$. Both firms have zero costs. The effect of the price of the good produced by Firm 2 in the demand for Firm 1 may be high or low. More specifically, b_1 takes 0 and 1 with probabilities 2/3 and 1/3, respectively. Further, $b_2 = 0$. Each firm knows its own parameter b_i, Firm 1 knows, in addition, b_2, but Firm 2 does not know b_1. All this is common knowledge.

(a) Which are the strategies for each firm?
(b) Compute the Bayesian-Nash equilibrium.

17. Calculate the pure strategy Bayesian-Nash equilibria of the following static Bayesian game. The payoffs of the players, *A* and *B*, are those of Game 1 or of Game 2, with the probability of each being equal. Player *A* is informed which of the games, 1 or 2, was chosen, but Player *B* does not know which of the games is being played. Player *A* chooses *x* or *y*; simultaneously, Player *B* chooses *m* or *n*.

Game 1

	m	*n*
x	1, 1	0, 0
y	0, 0	0, 0

Game 2

	m	*n*
x	0, 0	0, 0
y	0, 0	2, 2

18. A firm (Player 1) is already established in a market and must decide whether or not to construct a new factory. The potential benefits of this action depend on whether another firm (Player 2) enters or does not enter the market. Player 2 is uncertain of the cost faced by Player 1 of constructing the factory, which Player 2 believes may be high or low with probabilities 1/3 and 2/3 respectively. Player 1 knows the construction costs. The decisions to construct (*C*) or not to construct (*NC*) and to enter (*E*) or not to enter (*NE*) are taken simultaneously. The payoffs are given by:

High costs

	E	*NE*
C	0, -1	15, -1
NC	2, 1	3, 0

Low costs

	E	*NE*
C	1,5, -1	3,5, 0
NC	2, 1	3, 0

Find the Bayesian-Nash equilibria of this game.

19. Eva and Bernard are the only participants in a simultaneous auction (with sealed bid) for an object. The auction is conducted according to the following rules:

- Each player can only bid either 100 or 200 euros.
- Once they have made bids the object goes to the player that made the highest bid, and in case of a draw they toss a coin to see who gets it.
- The winner pays what he or she bid.

Both players know that Eva values the object at 300 euros. But Bernard's type is determined at random, and so only Bernardo knows his own valuation, which could be 150 euros or 250 euros. The probability of each type is 1/2. Calculate the Bayesian-Nash equilibria of this game.