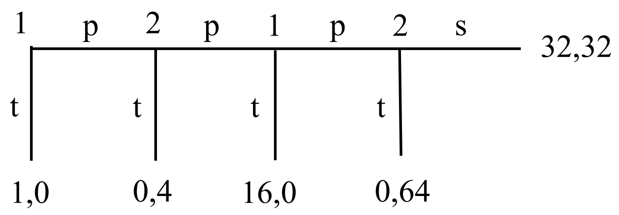
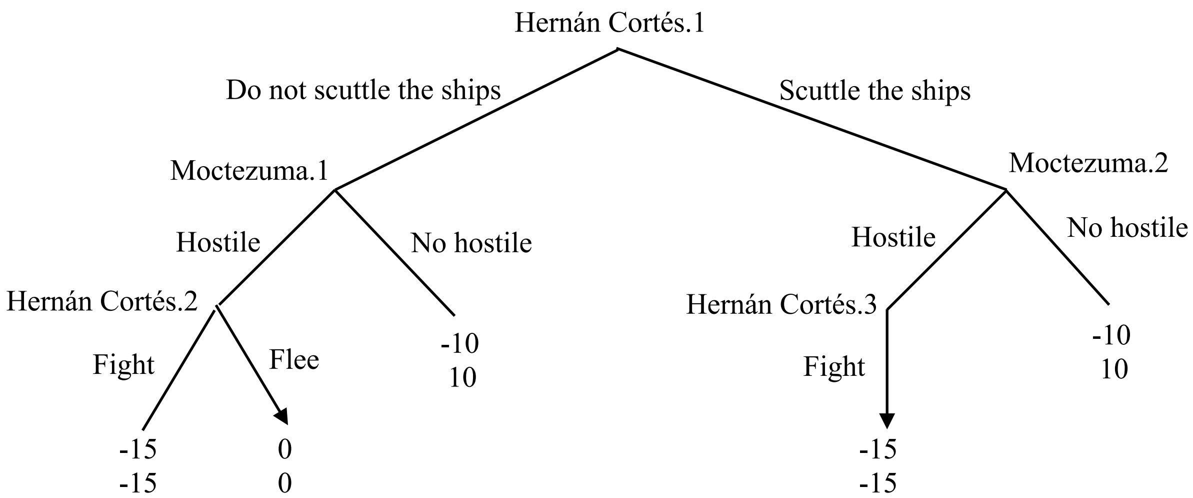
Universidad Carlos III de Madrid
GAME THEORY
Exercises on dynamic games

1. Consider the game in transparency 11 in “Dynamic Games 3. Credibility”, where the four-period centipede game is illustrated, and that we reproduce here for convenience.



(a) How many strategies does each player have?
(b) How many subgames are there? Identify them.
(c) Find the set of all strategies for each player.
(d) What strategy profiles result in the payoffs (16, 0)?
(e) Calculate the backward induction equilibria.

2. Consider the game in transparency 19 from “Dynamic Games 3. Credibility”, which illustrates the extensive form of the example of strategy elimination, and that we reproduce here for convenience.



(a) How many strategies does each player have?
(b) How many subgames are there? Identify them.
(c) Find the set of all strategies for each player.
(d) What strategy profiles result in the payoffs (0, 0)?
(e) Calculate the backward induction equilibria.

3. In Freedonia, a typical consumer values her health at 100. There is ¼ probability that her health deteriorates to a value level of 0. Thus, at the present time, her expected health value is 75. She can buy full insurance at the price of 20, which means that the policy will cover all the costs to recover her health completely. The government is considering offering help to individuals in need, but this is a basic service that will recover the consumer's health only to a level of 40. The government's service will be offered for free to the consumer, but it has an estimated cost of 5 per consumer for the government. The government's goal is to maximize the health of the typical consumer minus the cost to the government. If the consumer does not buy private insurance, she will make use of the government service if necessary. The government chooses first whether to go ahead with the health service or not. Then, the consumer chooses whether to buy private insurance after observing the government's choice.

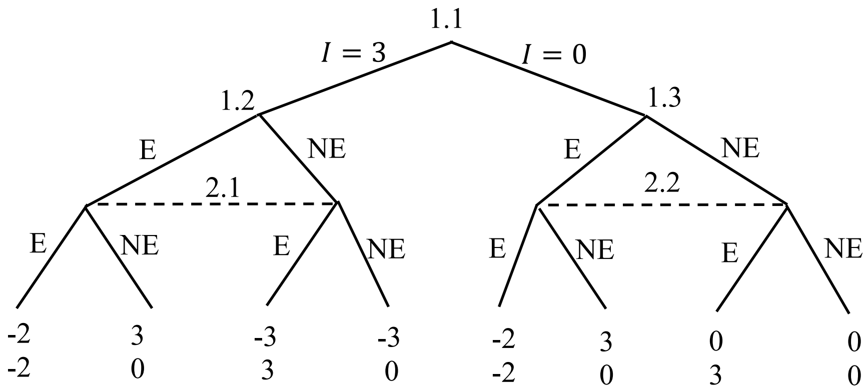
(a) Draw the extensive form of the game. Remember to carefully label all nodes and actions.
(b) Find the subgame perfect Nash equilibria.
(c) Find the associated normal form and the Nash equilibria. Compare with (b).

4. Repeat Problem 3 for the case where the consumer chooses first. Compare the equilibria between the two problems.

5. Merche and Antonio have to decide where to go on vacation. They have three options: Alicante (A), Barcelona (B) and Córdoba (C), but they do not reach an agreement where to go. In order to decide, they use the following mechanism. First, Merche vetoes one of the three places. Then, Antonio, after observing Merche's veto, vetoes another place. They go to the place that has not been vetoed. Merche prefers A to B and B to C, while Antonio prefers C to B and B to A. Assuming that each player assigns a utility of 3 to the favored place, a utility of 2 to the second best alternative and an utility of 1 to remaining city, and that both players want to go together on vacation, answer to the following questions:

(a) Represent the game in extensive and normal form.
(b) Find the Nash equilibrium/a in pure strategies.
(c) Which of the Nash equilibria previously found are subgame perfect Nash equilibria? Explain your answer. Where do Merche and Antonio go on vacation?

6. Consider the game in transparency 15 from "Dynamic Games 3. Credibility", which illustrates the extensive form of the example of a "costly commitment", and that we reproduce here for convenience.



(a) How many information sets does each player have? Find them.
(b) How many strategies does each player have? List them.
(c) How many subgames are there? Identify them.

(d) Find the sugbame perfect Nash equilibria.

7. Among the many adventures that Odysseus (Ulysses) endured during the ten years that took him to sail back to Ithaca after the Trojan War, his encounter with the sirens stands out (Homer, 8th century B.C.). Circe, a goddess seeking to protect Odysseus, warns him that he will be passing near the place inhabited by the Sirens, monsters that pretend to be beautiful women, and whose beguiling songs lure any man into jumping overboard to be killed by them. Odysseus wants to sail towards this place to hear the songs, but not to throw himself into the sea. He is also aware that his mad self would be willing to do just that.

(a) Represent this episode as a game between Odysseus-sane and Odysseus-insane. Use numbers -1,0 and 1 for payoffs after the outcomes. Note: If Odysseus does not listen to the Sirens, you may assume that the hypothetical payoffs for Odysseus-insane are the same as for Odysseus-sane.
(b) Find the subgame perfect Nash equilibria.

Knowing the above, Odysseus is still considering to listen to the Sirens. To this end he orders his men to tie him to the mast and put wax in their ears. Now he cannot jump over the board.

(c) Draw the game after being tied to the mast.
(d) Draw the subgame perfect Nash equilibria.
(e) Draw the whole game, that includes the decision to be tied to the mast or not, and find the subgame perfect Nash equilibria.

8. Anna and Bob want to meet next Saturday. They can either meet in a park, or in a coffee shop. Anna loves nature, while Bob prefers coffee. Anna receives a payoff of 3 if they meet in a park, and 1 if they meet in a coffee shop. Bob receives a payoff of 3 if they meet in a coffee shop, and 1 if they meet in a park. If they end up in different places, each receives a payoff of zero. At the beginning of the game, Anna chooses whether to call Bob to tell him about her plans. If she does, she can choose where to go (to the park, or to the coffee shop), and Bob chooses where to go (to the park, or to the coffee shop) after knowing her choice. If she does not call Bob, they choose where to go simultaneously. If Anna calls Bob, she has to pay an additional cost of 1 (hence, her payoffs are reduced by 1).

(a) Draw the game tree.
(b) How many subgames does the game have?
(c) How many strategies does each player have?
(d) Find the subgame perfect Nash equilibria in the subgames that begin after each of Anna's initial choices (to call Bob, or to not call Bob).
(e) Find all subgame-perfect Nash equilibria of the entire game.

9. Carlos and Natalia face the following situation. Natalia has to choose between two actions: *S* to stop playing with Carlos, or *C* to continue playing with him. In case she chooses *S*, she gets a payoff equal to *y*. In case she chooses *C*, they will have to play a simultaneous game where Natalia chooses between *U* and *D* while Carlos chooses between *L* and *R*. The payoff matrix for the simultaneous game is:

	L	R
U	3, 1	2, -1
D	1, 0	4, 5

(a) Find all the Nash equilibria for the simultaneous game that starts after Natalia chooses *C*.
(b) Find Natalia's payoffs in each of the Nash equilibria from part (a).
(c) Find all possible values for y such that Natalia's first action is always *C* in all and each of the subgame perfect Nash equilibria.
(d) List all the subgame perfect Nash equilibria for such values of y.

10. Three countries (A, B and C) must decide whether to declare war on D. If two or three do, they will win without any problem and will be rid of a common enemy, something that they each value at 100. Winning the war costs 120, a quantity that will be divided equally among the countries that declare war. If only one declares war, nothing will happen.

(a) Find the Nash equilibria in pure strategies if countries decide simultaneously.
(b) Find the subgame perfect Nash equilibria if the decision is sequential, first A, then B and C, in that order, and if each one knows what the previous country has decided.
(c) Find the equilibrium paths for each of the subgame perfect Nash equilibria.

11. Two firms compete in prices in a market with the demand detailed below. Firm 1 decides first announcing a price, and then Firm 2 decides on a price also, having observed Firm 2's decision. Prices can only be integer numbers between 1 and 10. Consumers will buy from the firm that puts the lowest price according to the demand. If both firms put the same price, they will share the market equally. Both firms incur a unitary production cost of 3.

P	1	2	3	4	5	6	7	8	9	10
Q	90	80	70	60	50	40	30	20	10	0

(a) Find the subgame perfect Nash equilibria. Note: no need to draw the game. For each of the different prices Firm 1 can set, just find which is the best reply by Firm 2 and work with that.

A CEO in Firm 1 proposes a new marketing strategy. In addition to putting its price, Firm 1 will publish a compromise by which it will match the rival's price in case this one puts a lower price. According to the country's law, this kind of public compromise is binding, so if the firm does not honor its compromise, it will pay a penalty of 100. The consumers' association states that this strategy is contrary to competition. The firm argues that it guarantees a low price and therefore favors competition.

(b) Who is right? To answer this question, find the new subgame perfect Nash equilibria and compare the result with the subgame perfect Nash equilibria in (a). Note: as before, no need to draw the game. Just find the new best replies.

12. Two fishing boats work in a lake. Each one of them can be greedy or conscious. If both are conscious, each can capture fish worth 500. If both are greedy, they will exhaust the resource and captures will be worth 100 for each ship. If one is greedy and the other one is conscious, their captures will be worth 600 and 50, respectively. The decision about how much to fish is done simultaneously and independently.

(a) Find the Nash equilibria. Are they efficient?

Being aware of the bad results in the equilibrium, they resolve to sign a contract stipulating that the greedy ship will pay a penalty of 300. Thus, in a first moment, the owners of the ships must simultaneously decide between signing the contract or not. If they both sign, the contract is valid and can be lawfully enforced. If either of them does not sign, the contract is not valid and the game will be as in (a).

(b) Find the subgame perfect Nash equilibria in pure strategies.
(c) Repeat the exercise in case they make the decision about signing the contract sequentially, with the second ship knowing what the first one did.

13. In Problem 17 in static games, consider now that Firm 1 plays first and that, then, firms 2 and 3 play simultaneously after observing Firm 1's choice. Find the subgame perfect Nash in the same both cases as in the original exercise.

14. Suppose that *Extra* and *Ultra* are the only car-producers that compete in the Spanish car market. The demand for cars in Spain is given by $p(Q) = 10 - Q$, where Q is the total quantity produced by the two car-producers, Q = q_E +q_U. The total costs faced by *Extra* and *Ultra* are given respectively by $C_E(q_E) = 3q_E$ and $C_U(q_U) = 2q_U$.

(a) Assume that *Extra* and *Ultra* choose simultaneously the quantities that they will produce, q_E and q_U respectively. Determine the reaction function of each car-producer and the Nash Equilibrium of this game. Compare the equilibrium profits of the two car-producers.
(b) Assume now that the game changes. In the new game, *Extra* chooses its quantity q_E first. *Ultra* chooses its own quantity q_U after observing the decision of *Extra*. Represent this game in extensive form. Find the Subgame Perfect Nash Equilibrium (SPNE) of this game.
(c) What is the amount of money that *Ultra* would have to pay to *Extra* in order to make *Extra* agree to choose its quantity simultaneously with *Ultra*? Is *Ultra* willing to pay this amount of money?

15. Two firms, Peurot and Fol, compete in the market for cars, where demand is $P(Q) = 2 - Q$ and the state of technology is such that marginal cost is $c = 1$. In this market, Peurot is the leader (chooses first) and Fol is the follower (chooses its quantity knowing the leader's chosen quantity). Fol cares not only about its own profit but also the volume of sales, since it must capture some market share, as long as it does not lose money. Specifically, the utility function for Fol is given by

$$U_F(q_P, q_F) = \alpha\Pi_F(q_P, q_F) + (1-\alpha)q_F \text{ if } \Pi_F(q_P, q_F) \geq 0$$

(and $\Pi_F(q_P, q_F)$ otherwise)

whereas the leading firm only cares about own profit, i.e.,

$U_P(q_P, q_F) = \Pi_P(q_P, q_F)$.

(a) If $\alpha > 1/2$, calculate the subgame perfect Nash equilibria of this game. Determine for what values of α the leader produces more than the follower in the equilibrium.
(b) Now suppose that $\alpha = 0$. Calculate the subgame perfect Nash equilibria.

16. Three neighbors $\{A, B, C\}$ must agree to repair a common zone. The best budget implies a cost of 210 and each neighbor values the project at 100.

(a) Assume that each neighbor decides voluntarily how much to pay and that the decisions are made sequentially (first *A* decides, then *B* and, finally, *C*) so that when a neighbor makes her decision she is aware of the decisions made by those before her. The project is undertaken if the voluntary contributions are enough to pay for the project. Which are the subgame Nash equilibria?
(b) Now assume that the neighbors simply vote either Yes or No. The project goes ahead only if all say Yes. Also, they know that if they go forth with the project they will pay the cost in equal shares. Find the subgame perfect Nash equilibria if the vote is sequential as in (a).

17. In the agenda manipulation game (class notes in the course web page, slide #22 in Dynamic applications), show which voting ordering the president of the committee will decide if she wants alternative C to win.

18. In the following bargaining game, a firm (*F*) and a workers' union (*U*) have to share the benefits generated by their economic activity. Assume that the benefits are equal to 2 million euros. The game has three stages, and the players alternate making offers, starting with the firm. Once an offer is accepted, the game ends. If the players do not reach any agreement, after the third proposal, both of them get a zero payoff.

(a) What will be agreement reached in equilibrium and in which time period will the agreement be reach if the discount factor of both players is $\delta = 1/4$?
(b) What will be agreement reached in equilibrium and in which time period will the agreement be reach if *F* has a discount factor $\delta_F = 1/4$ and *S* has $\delta_S = 1/2$?
(c) Compare the two agreements and try to provide an intuitive argument to support the results you have found.

19. A conservative country has to decide whether or not to pass a legislation to legalize some social rights. To clarify ideas, think of the decision to be made following negotiations by the two main groups in the country, the NO group and the YES group. Say that the NO group feels very strongly about its cause and has preferences represented by the utility function $u_{NO}(x) = (1 - x)^3$, where $x \in [0,1]$ measures the permissibility of the legislation ($x = 1$ means equal rights for all, $x = 0$ is the status quo, and intermediate values mean different degrees of equality). The utility of the YES group is $u_{YES}(x) = x$. Negotiations are conducted as follows: first, the NO group makes an offer (a value of $x \in [0,1]$), and then the YES group either accepts (A) or rejects (R). If the offer is accepted, then it becomes the law. If it is rejected there will be social turmoil and confrontation that will end with an equal probability of victory for either of the two groups. In this case, the winner imposes its preferred legislation. The NO group does not care for social confrontation, while the YES group would see its utility reduced by C if it were to happen. Thus, its utility will be $u_{YES}(x) = x$ if there is no confrontation, and $u_{YES}(x) = x - C$ if there is.

(a) Draw the extensive form game in the usual way for bargaining problems.
(b) Find the subgame perfect Nash equilibria if $C = 0.2$.
(c) Find the maximum value of C for an agreement to be the outcome of a subgame perfect Nash equilibrium.

20. In the example for the Coase theorem (class notes in the course web page, slides #12, 13 and 14, Dynamic Negotiation), consider the case where the physician has the right to ask the baker to leave, but not until they finish the negotiations, that last two periods of alternating offers, where the baker makes the first offer and the discount factor for both is $\delta = 0.6$.

(a) Draw the extensive form game.
(b) Find the subgame perfect Nash equilibria.

Extra Exercises

21. Three neighbors (Ana, Bea, Cruz) have to choose one among three projects (*a*, *b*, *c*). Preferences are represented in the following table. Each column represents the order of preferences of the corresponding neighbor, the preferred project being located above in each column.

Ana	Bea	Cruz
a	*b*	*c*
b	*a*	*a*
c	*c*	*b*

The choice is realized using a simple majority rule in a two step vote. In the first step, the neighbors choose between *a* and *b*, and the winner of this step competes against *c*. From this second step is selected the project that will be implemented.

(a) What would be the result if, in each step, preferences are truly revealed? (i.e., they vote for the project they prefer).

We analyze now this election mechanism as a game (the neighbors can vote strategically)

(b) Assume that *a* has been chosen in the first step. Explain why the fact that all neighbors vote for *c* in the second stage is a Nash equilibrium.
(c) Why isn't this equilibrium very plausible? Which refinement (or selection criterion) would eliminate this equilibrium?
(d) Which subgame perfect Nash equilibrium would satisfy this refinement in both steps?

22. Two partners *A* and *B* are trying to finish a project. Each of them will receive 25 million euros when the project is completed, but nothing before its completion. The project needs 7 millions in order to be complete. None of the partners can commit in a credible way to pay such a sum in the future, so they decide the following: In a first stage, Partner *A* chooses to contribute with c_A. If this quantity is enough to complete the project, the game is over and each partner receives the 25 millions. In case it is not enough (c_A is less than 7 millions), Partner *B* chooses its contribution c_B. If the sum of both contributions allows completing the project, each of them receives the 25 millions, if not, they do not receive anything. The only way to get the money and contribute to the project is to remove it from the other activities of both partners. We assume that from these activities, each partner can obtain c_i^2 , (i = A,B).

(a) Find the subgame perfect Nash equilibrium.
(b) Assume now that the sum that is missing in order to complete the project is 12 millions. Find the new subgame perfect Nash equilibrium.

23. Two firms, Peurot and Fol, compete in the market for cars, where demand is $P(Q) = 2 - Q$ and the state of technology is such that marginal cost is $c = 1$. In this market, Peurot is the leader (chooses first) and Fol is the follower (chooses its quantity knowing the leader's chosen quantity). Fol cares not only about its own profit but also the volume of sales, since it must capture some market share. Specifically, the utility function for Fol is given by

$$U_F(q_P, q_F) = \alpha\Pi_F(q_P, q_F) + (1 - \alpha)q_F,$$

whereas the leading firm only cares about own profit, i.e.,

$$U_P(q_P, q_F) = \Pi_P(q_P, q_F).$$

(a) If $\alpha > 1/2$, calculate the subgame perfect Nash equilibrium of this game. Determine for what values of α the leader produces more than the follower in the subgame perfect Nash equilibrium.
(b) Now suppose that $\alpha = 0$. Calculate the subgame perfect Nash equilibrium.

24. Two firms produce differentiated goods, for example, Dell and Acer. Each of them chooses its price in order to maximize its own profits. Let p_1 be the price of Firm 1 and p_2 be the price of Firm 2. Given prices p_1 and p_2, Firm 1 will be able to sell $q_1 = 100 - p_1 + 0{,}5p_2$ and Firm 2 will be able to sell $q_2 = 100 - p_2 + 0{,}5p_1$. Both firms have marginal costs of 50.

(a) Suppose the two firms move simultaneously. Find their best response functions. Find out the Nash equilibrium of this game. Also find each firm's profits at the equilibrium prices.
(b) Suppose now Firm 1 moves first, and Firm 2 observes Firm 1's choice of p_1 before choosing p_2. Find the prices, quantities and profits in the subgame perfect Nash equilibrium.
(c) Is there an advantage of moving first in this game?

25. Ester and Fernando play a game where each of them has to choose a number from the interval [0, 1]. First Ester writes a number x, $x \in [0, 1]$. Then, after observing x, Fernando chooses a number y, $y \in [0, 1]$. Ester's and Fernando's utility functions are $U_E(x, y) = \min(x, y)$ and $U_F(x, y) = (2x - y)^2$ respectively.

(a) Represent this game in the extensive form, indicating if it is a perfect or imperfect information game, how many information sets each player has, and how many subgames the game has.
(a) What are Fernando's best responses for each of the following Ester's choices: $x = 0$, $x = 1/4$, $x = 1/2$ and $x = 1$?
(b) Find the subgame perfect Nash equilibrium for the game. Suppose that, in case of being indifferent between two numbers, Fernando always chooses the greater one.
(c) Find the utilities obtained by Ester and Fernando in the subgame perfect Nash equilibrium.

26. Consider the following game, where Jorge chooses between the actions *A* and *B*, while Alicia chooses between *C* and *D*:

	C	*D*
A	2, 1	0, 0
B	0, 0	1, 2

(a) Find all the Nash equilibria in pure and mixed strategies for this game. Indicate the utility each of the players gets in each of the equilibria.
(b) Suppose now that Jorge and Alicia choose their actions sequentially, with Jorge choosing first, and Alicia being able to observe Jorge's action before choosing her own action. Find ALL the subgame perfect Nash equilibria for this new game. How much gets each of the players in these equilibria?

27. Consider a firm (*F*) that selects the number of workers $L \geq 0$ and a union (*U*) that fixes wages, $w \geq 0$. Firm's profits are given by $\Pi_E(w, L) = 100L - 0{,}1L^2 - wL$ whereas union's payoff is given by total wage, $\Pi_S(w, L) = wL$. Suppose that the union chooses first the wage *w*, and the firm observes *w* and then chooses labor input *L*.

(a) Draw the extensive form of the game and find the subgame perfect equilibrium.
(b) Suppose that the union is worried about reaching an employment level of X at least, so that its payoff function is now

$$\Pi_S(w, L) = (L - E)w$$

Draw the extensive form of the game and find the subgame perfect equilibrium, as a function of *E*.

28. Consider the following effort-negotiation game between two partners in a joint project X. In a first stage partners 1 and 2 must choose simultaneously the effort level, $e_i \in [0,\infty)$ for i = 1, 2, to exert in the joint project *X*. Gains from project *X* are:

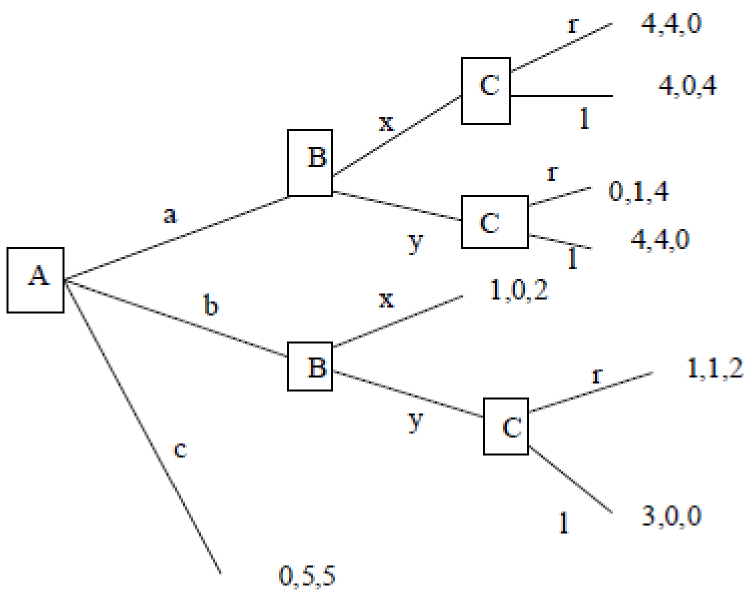
$$\Pi(e_1, e_2) = e_1 + e_2 + e_1 e_2.$$

The cost of exerting effort is given by

$$C(e_i) = \frac{1}{2} e_i^2, \text{ for } i = 1,2.$$

In a second stage, once e_1 and e_2 have been chosen, the two partners agree to share those gains as follows. They flip a coin and if the result is heads Partner 1 proposes a division of the gains between him and Partner 2. The latter must decide whether to accept or reject that division. If rejected, the game ends and both partners earn zero. If tails, the allocation procedure is the same but Partner 2 will be the one proposing the division of gains. Find all the subgame perfect equilibria of the game. Specify your results in terms of the strategies of each of the players

29. Consider the following Extensive form game among three players



Game problem 29

(a) If the game is one of perfect information and all players can observe previous actions of the other players, determine all subgame perfect Nash equilibria in pure strategies, the equilibrium path and the equilibrium payoffs.
(b) Assume now that Player *B* cannot observe the actions of *A*. In this case:
  (i) Represent the new game in extensive form.
  (ii) Are there pure strategy subgame perfect Nash equilibria?
(c) Suppose now that *A*'s actions are observable by *B* and *C*, but that *C* cannot observe the actions of *B*. In this case:
  (i) Represent the new game in extensive form.
  (ii) Are there any pure strategy subgame perfect Nash equilibria?

Note: If a player does not participate in a given subgame, his/her payoffs are not relevant for the calculation of the NE in that subgame

30. Two companies competing in a market are considering the possibility of a mail sale campaign. The cost of this campaign is 200 euros. If only one company makes the campaign, it will ensure for itself the sale of 18 units at a price of 30 euros, while if the two companies do it, everyone would sell 12 units at a price of 15 euros. Both companies can produce any quantity at zero cost. Suppose that firms, after deciding whether or not to undertake the mail sale campaign, compete in quantities in a market whose demand is given by the function $P = (99 - V) - q$, where P represents price, V is the total volume of sales and $q = q_1 + q_2$ is the total amount sold in this market (net of mail sales). Consider the sequential game in which the two companies must first decide simultaneously whether to make the mail sales campaign or not to do it, and, after knowing the decisions about the campaign, there is Cournot competition.

(a) Draw the extensive form game
(b) Show the information sets for each company. Show also the subsets of this game.
(c) Compute the Nash equilibria of all subgames.
(d) Find all the SPNE of the game.