Bayesian games

Lesson 1: Types, Beliefs and Bayesian Equilibrium

Universidad Carlos III de Madrid
Recall: static game with complete information

- Players
- Strategies (actions)
- Payoffs or preferences for each strategy combination
- All of the above is *common knowledge* among the players.
The Bayesian game

- Payoffs are not common knowledge

- Incomplete Information means that at least one player does not know someone else’s payoffs.

- Static game with incomplete information = Static bayesian game.
Examples

- Cournot Duopoly, with firms not knowing rivals’ costs.
- Auction in which bidders don’t know others’ valuations.
- Private contributions to a public good where individuals don’t know others’ valuations.
- A bargaining process in which the other’s discount factor is unknown.
- Battle of the sexes when one doesn’t know if the other prefers to be alone or go in a date.
In this class we’ll learn to:

- Identify the elements of an incomplete information game and represent them in a game form.
- Understand a Bayesian game as an extensive form game with imperfect information.
- Find Bayesian Nash Equilibria (BNE).
Example 1

- **Player 1** can choose between actions A and B.
- **Player 2** can choose between actions I and D.
- Payoffs depend on the players’ types.
- **Player 1 can only be of one type** which is known by Player 2.
- **Player 2 may be of types x and y.**
- Player 2 knows his type, but Player 1 does not know Player 2’s type (asymmetric incomplete information).
- Player 1 knows that Player 2 is of type x with **probability** 2/3, and of type y with probability 1/3.
Model “not knowing the payoffs” as “not knowing the types”

2 type x (2/3)

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4, 3</td>
<td>3, 1</td>
</tr>
<tr>
<td>B</td>
<td>3, 6</td>
<td>2, 3</td>
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</table>

2 type y (1/3)

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<th></th>
<th>I</th>
<th>D</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td>3, 3</td>
<td>1, 6</td>
</tr>
<tr>
<td>B</td>
<td>1, 1</td>
<td>5, 3</td>
</tr>
</tbody>
</table>
Bayesian Game as a Dynamic Game of Incomplete Information

Player 1 has one information set, thus his strategies have one action. Player 2 has two information sets, and her strategies are: II, ID, DI, DD.
Player 2’s best reply

- Best reply correspondence.
- Player 2 knows her type (and the type of Player 1):
  - If Player 2 is of type $x$:
    - Strategy D is strictly dominated by strategy I. Her best reply action is I.
  - If Player 2 is of type $y$:
    - Strategy I is strictly dominated by strategy D. Her best reply action is D.
Player 1’s best reply

- Player 1 knows his type, but not the type of Player 2.
- Player 1 calculates his expected payoff if he plays A and his expected payoff if he plays B for all possible strategies by Player 2, S2={II, ID, DI, DD}

<table>
<thead>
<tr>
<th></th>
<th>II</th>
<th>ID</th>
<th>DI</th>
<th>DD</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td><strong>11/3</strong></td>
<td>3</td>
<td>3</td>
<td><strong>7/3</strong></td>
</tr>
<tr>
<td>B</td>
<td>7/3</td>
<td><strong>11/3</strong></td>
<td>5/3</td>
<td>3</td>
</tr>
</tbody>
</table>
Bayesian Nash Equilibrium

- Given that Player 2 has dominant strategies, she plays $I$ if she is of type $x$ and $D$ if she is of type $y$.
- Given strategy $ID$, the best reply for Player 1 is $B$.

The only Bayesian equilibrium of this game is $(B, ID)$. 
Representation of a Bayesian game

- Player set: $N = \{1, 2, \ldots, n\}$.
- Players’ *types*.
- Probability distributions over combinations of types, *(a set of beliefs over rivals’ types)*
- Possible actions/strategies.
- Payoff functions that now depend not only on actions, but *also on types*. 
Payoffs, beliefs and strategies

- Player $i$’s **payoff function** is written as:

  \[ u_i(a_i, a_{-i}; t_i, t_{-i}) \]

  where $a_i \in A_i$, $a_{-i} \in A_{-i}$, $t_i \in T_i$, $t_{-i} \in T_{-i}$.

- **Beliefs:**
  - Each player knows his type and, thus, his payoff function.
  - Each player that does not know the payoffs of some rival, has some beliefs (a probability distribution) over their types. We’ll denote these beliefs by

    \[ p_i(t_{-i} | t_i) \]

    for $t_{-i} \in T_{-i}$, $t_i \in T_i$.

- **Strategies:** One action for every possible type of the player.
In example 1

- Players, N={1,2}.
- Players’ *types*: Player 1 has one type, and Player 2 has two: x, y.
- Probabilities about types: Each one of the three types has beliefs about the other players.
  
  \[
  (p(t_2 = x / t_1) = 2/3, p(t_2 = y / t_1) = 1/3).
  \]
  
  \[
  (p(t_1 / t_2 = x) = 1).
  \]
  
  \[
  (p(t_1 / t_2 = y) = 1).
  \]

- Strategies for 1: {A, B}. For 2: {II, ID, DI, DD}
- Payoffs: the 2 matrices (slide 7).
The Battle of the Sexes with incomplete information

- A couple: she is a football fan and he loves opera.
- His preferences depend on whether he is stressed or not. If he is stressed he prefers to spend the night alone. If he is calm (normal) he prefers opera rather than football, and he prefers the company of his girlfriend rather than being alone.
- She believes that it is equally likely that he is stressed and that he is calm.
Payoffs

Payoffs if HE is calm
Prob. = 1/2

<table>
<thead>
<tr>
<th>SHE</th>
<th>F</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>O</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Payoffs if HE is stressed
Prob. = 1/2

<table>
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<tr>
<th>SHE</th>
<th>F</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2, 0</td>
<td>0, 2</td>
</tr>
<tr>
<td>O</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
</tbody>
</table>
Best reply by HE

<table>
<thead>
<tr>
<th></th>
<th>F</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHE F</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>SHE O</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

BR by HE (F) = FO    BR by HE (O) = OF

If she chooses football the best reply by HE is: football if calm and opera if stressed.

If she chooses ópera the best reply by HE is: opera if calm and football if stressed.
Best reply by SHE

<table>
<thead>
<tr>
<th></th>
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<th>FO</th>
<th>OF</th>
<th>OO</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>O</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>

- $\text{BR(FF)} = F$
- $\text{BR(FO)} = F$
- $\text{BR(F)} = FO$
- $\text{BR(O)} = OF$
- $\text{BR(OF)} = F$
- $\text{BR(OO)} = O$

- $\text{BNE: (football, (football if calm, and opera if stressed))}$
The Battle of the Sexes with incomplete information. Alternative analysis.

Payoffs by HE
1 0 0 2 0 1 2 0

Payoffs by SHE
2 0 0 1 2 0 0 1
SPNE in the extensive form

- There are no subgames, thus the SPNE coincides with the NE

<table>
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<th>OF</th>
<th>OO</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>2, 0.5</td>
<td>1, 1.5</td>
<td>1, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>O</td>
<td>0, 0.5</td>
<td>0.5, 0</td>
<td>0.5, 1.5</td>
<td>1, 1</td>
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Extensions

- SHE has two types, HE has only one. The matrix of the normal form is 4x2.
- SHE has only one type. The matrix of the normal form is 2x8 (we will not see this).
- SHE and HE have two types each (we will not see this).