Dynamic Games

2: Imperfect information

Universidad Carlos III de Madrid
Dynamic Games with Imperfect Information

- Games in which at least one of the following happens:
  - A player does not know which action some other player has taken.
  - Some players have different information over a result of a nature move.
- This translates into the fact that some players don’t know with certainty in which one of their nodes actually are at some point in the game.
- The nodes a player cannot tell apart are nodes in which the player has the same information. Each set of nodes in which this occurs is called an information set.
- Trivially, when a player knows that she is a node, that node is an information set of one element.
- Graphically, we will join the nodes belonging in an information set with a dotted line or a “cloud”.
Example: dynamic battle of the sexes

- Player 2 **knows** what player 1 did.
- Player 2 **does not know** what player 1 did.
- Nodes 2.1 y 2.2 belong in an information set.
- There is no backward induction equilibrium, but there are SPNE.
A complicated example

• Information sets:
  • Player 1: \{1.1\}, \{1.2, 1.3\} y \{1.4\}.
  • Player 2: \{2.1\}, \{2.2\}, \{2.3, 2.4\} y \{2.5\}.
  • Player 3: \{3.1\} y \{3.2\}.

• Static and dynamic subgames:
  • At 2.1 begins a static (sub)game.
  • At 2.2 begins a dynamic (sub)game.
  • At 3.1 begins a (sub)game with characteristics of both kinds of game.
Extensive form, normal form and subgames

• We have to add or change the following in the extensive form definition for games of imperfect information:
  • Group the nodes of a player in information sets.
  • Define actions in each information set (not in each node): informally, an action implies choosing the same edge in each node of a given information set.

• To obtain the normal form, it is enough to define a player’ strategy as a vector that defines an action in each information set (rather than in each node).

• Subgames are defined as before, but with a new rule “do not break information sets”.
A complicated example

- Which subgames are there?
  - Those starting at 1.1, 2.1, 2.2, 3.1, 3.2, 1.4 and 2.5.

- Which is the set of strategies for Player 1?
  - \{(I,a,x), (I,a,z), (I,b,x), (I,b,z), (C,a,x), (C,a,z), (C,b,x), (C,b,z),
  (D,a,x), (D,a,z), (D,b,x), (D,b,z)\}.
  - Example: \((C,a,z)\) in red.
• Which subgames are there?
  • Those starting at 1.1, 3.1, 3.2, 1.4 y 2.5.

• Which is the set of strategies for Player 2?
  • \{ (I,a,r), (I,a,t), (I,b,r), (I,b,t), (D,a,r), (D,a,t), (D,b,r), (D,b,t) \}.
  • Example: (D,a,r) in red.
Example to find SPNEa

• Player 1 chooses between A and B.
• If he chooses A, he and Player 2 play the chicken game.
• If he chooses B, they play the battle of the sexes.

We have numbered the information sets (rather than the nodes).
Which subgames are there?
Three, starting at 1.1, 1.2 and 1.3.
Begin by solving 1.2 and 1.3.
Example to find SPNEa

• To simplify, we only consider pure strategies.

• The subgame starting at 1.2 is the chicken game with NE in pure strategies: \((K, S)\) and \((S, K)\).

• The subgame starting at 1.3 is the battle of the sexes with NE in pure strategies: \((F, F)\) and \((O, O)\).

• To find the equilibrium action at 1.1, we must consider four possibilities:

<table>
<thead>
<tr>
<th></th>
<th>1.2: (K, S)</th>
<th>1.2: (K, S)</th>
<th>1.2: (S, K)</th>
<th>1.2: (S, K)</th>
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<tbody>
<tr>
<td></td>
<td>1.3: (F, F)</td>
<td>1.3: (O, O)</td>
<td>1.3: (F, F)</td>
<td>1.3: (O, O)</td>
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<tr>
<td>A</td>
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<td>3</td>
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<td>3</td>
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</tr>
</tbody>
</table>

1.1 prefers A

1.1 prefers A

1.1 prefers B

1.1 is indifferent

SPNE: \(((A,K,F), (S,F))\)

SPNE: \(((A,K,O), (S,O))\)

SPNE: \(((B,S,F), (K,F))\)

SPNEa: \(((A,S,O), (K,O))\) and \(((B,S,O), (K,O))\)
Two ways to write the SPNEa

• The canonic way: sort by players.
• The convenient way: sort by subgames.
• In the example before, the equilibrium \(((A,K,F), (S,F))\) is written in the canonic way.
• The convenient way is: \((A, (K,S), (F,F))\).

By players:

\[\begin{align*}
((1.1, 1.2, 1.3), (2.1, 2.2)) \\
((A, K, F), (S, F))
\end{align*}\]

By subgames:

\[\begin{align*}
(1.1, (1.2, 2.1), (1.3, 2.2)) \\
(A, (K, S), (F, F))
\end{align*}\]
Example 2 on how to find ENPS

Formula 1 Game

• Before deciding what type of tires to use, Al can make a strategic maneuver that would prevent Ham from participating in the race.

• Thus, in a first stage, Al must decide whether to prevent or not Ham’s participation in the race (decisions P and NP).

• If Al prevents the participation of Ham, Al will have 4 points at the end of the race, and Ham will have none.

• If Al does not prevent Ham’s participation, both pilots must choose simultaneously the type of tires (rain or dry), with the results shown next.
Example 2 on how to find ENPS

Formula 1 Game

Subgames:
starting at AL.1 and
starting at AL.2
**Example 2 on how to find ENPS**

**Formula 1 Game**

- **Start by solving subgame at AL.2 after NP:**

  The normal form is:
  
  $\begin{array}{c|cc}
  & R & D \\
  \hline
  HAM.1 & 1,2 & 2,3 \\
  AL.2 & 5,4 & 0,3 \\
  \end{array}$

- **NE** = \{(D, R), (R, D), (1/2[R]+1/2[D], 1/3[R]+2/3[D])\}

- **Payoffs** in NE of subgame after NP for AL: 5, 2 and 5/3, respectively.

- If AL.1 plays P he will get 4. Thus, if at the subgame after NP the NE is (D, R), he will choose NP. For any other NE he will choose P.

- Hence: **SPNE** : {((NP,D), R), (P,R), D), ((P,1/2), 1/3)}. 