Dynamic Games

Lesson 2: Dynamic games with Imperfect Information

Universidad Carlos III
DG with Imperfect information (DGII)

- Some player does not know the action taken by other player

- When a player does not know in which of two or more nodes she is we’ll say that they belong in the same information set (IS).

In a DGII, an IS may contain any number of nodes.

In a graph we just link with a dotted line the nodes that belong to the same information set.
Recall the game in Example 1:

- If Player 2 chooses knowing the action of Player 1 then we have a DGPI.

- If Player 2 chooses without knowing the action of Player 1 then we have a DGII. In this case, this game is equivalent to a static game. Denote this game by Game 1-bis.
DGII, Game1-Bis
Strategies in a DGII

- In a situation of imperfect information players cannot condition their action on the node they are, only on the information.

- For the model to make sense the number of branches after all nodes in a given information set must be the same. (If not, the player can get some information just by counting the number of options.)

Definition of strategy. A player’s strategy specifies an action in each of his information sets. (I.e., the player must choose the same action in all nodes that belong to the same information set.)

- Game 1-bis: Strategies of 1: {I, D}. Strategies of 2: {I, D}. Compare with Game 1, where Player 2 was able to condition his actions in two different information sets and, thus, had 4 strategies.
Solution Concepts

- It is not always possible to use backward induction in a game of imperfect information. (E.g.: in Example 1-bis which action maximizes the payoffs for Player 2?)

- Before solving Player 2’s problem one must solve Player 1’s. But Player 1 cannot know her best action until she knows what Player 2 is going to do.

- The concept of Subgame Perfect Nash Equilibrium (SPNE) allows us to consider both players’ problems simultaneously and offer a solution.

This is our solution concept for DGII.
Subgames

Subgame. The definition is the same as with perfect information (a part of the original game that maintains the structure of a game), but adding two more conditions:

• The initial node of the subgame must be a node that is the only node in an information set.)

• If a subgame contains a node, it must also contain all other nodes in the same information set as that node.

Intuitively, these conditions mean that subgames cannot brake the information sets of the original game.

In Game 1-bis there is only a subgame, beginning with Player 1. To consider any other node as an initial node for a subgame would mean to brake an information set of Player 2. Since the only subgame is the original game, the set of SPNE in pure strategies coincides with the set of NE in pure strategies: \{ (I,I), (D,D) \}. 
Example 2

Take the following extensive form game

```
  I_1  D_1
     /
    /
I_2  D_2  I_2  D_2
  3   -1  0   10
  5   -1  0   2
```

The corresponding normal form is:

```
<table>
<thead>
<tr>
<th></th>
<th>I_2</th>
<th>D_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>I_1</td>
<td>3, 5</td>
<td>-1,-1</td>
</tr>
<tr>
<td>D_1</td>
<td>0, 0</td>
<td>10, 2</td>
</tr>
</tbody>
</table>
```

Where one easily obtains $\text{NE}(G) = \{(I_1, I_2), (D_1, D_2)\}$. 
Assume now that Player 1 observes Player 2’s decision before taking her own.

The game has 3 subgames (the first payoff corresponds to the player playing first.)
Example 2

• The normal form is:

<table>
<thead>
<tr>
<th></th>
<th>$l_1i_1$</th>
<th>$l_1d_1$</th>
<th>$D_1i_1$</th>
<th>$D_1d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_2$</td>
<td>5, 3</td>
<td>5, 3</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
<tr>
<td>$d_2$</td>
<td>-1, -1</td>
<td>2, 10</td>
<td>-1, -1</td>
<td>2, 10</td>
</tr>
</tbody>
</table>

• $\text{NE}(G) = \{(l_2, l_1i_1), (l_2, l_1d_1), (d_2, D_1d_1)\}$

• Since this is a perfect information game, we can find the SPNE by backward induction:
  - In 1.1 Player 1 chooses $l_1$ since $3 > 0$.
  - In 1.2 chooses $d_1$ ($10 > -1$).
  - If 2 must choose between $l_2$, with payoff 5, and $d_2$, with payoff 2, he will choose $l_2$.

• $\text{SPNE}(G) = \{(l_2, l_1d_1)\}$
Example 3

Formula 1 game.

Before deciding what type of tires to use, Al can make a strategic maneuver that would prevent Ham from participating in the race.

Thus, in a first stage, Al must decide whether to prevent or not Ham’s participation in the race (decisions I and NI).

If Al prevents the participation of Ham, Al will have 4 points at the end of the race, and Ham will have none.

If Al does not prevent Ham’s participation, both pilots must choose simultaneously the type of tires (rain or dry), with the results shown next.
Strategies and Subgames

Strategies

- **AI:** \{ IL, IS, NIL, NIS \}
- **HAM:** \{ L, S \}

Subgames: There are 2 subgames:

- Subgame that starts at AL.1 (the whole game).
- Subgame that starts at AL.2.
Compute the NE in all subgames.
The subgame that starts at en AL.2 is a static game:
Subgame that starts AL.2 Players must play a NE.

```
  Ham
   L   S
  Al L  1,2  2,3
    S  5,4  0,3
```

NE in pure strategies: \{ \{(S, L), (L, S)\}\}.

NE in mixed strategies: \{ \{(1/2[L]+1/2[S], 1/3[L]+2/3[S])\}\}

Al receives in each of the equilibria the expected utility of 5, 2, and 5/3, respectively.
The Whole Game

First stage.

• If the equilibrium in the subgame is \((L, S)\) or the equilibrium in mixed strategies, Al prefers to prevent Ham from racing, as Al gets less than the 4 points he can get form playing I. Then \((I, (L,S))\), and \((I, (1/2, 1/3))\) are SPNE.

• If the equilibrium in the subgame is \((S, L)\), then Al prefers NI as he gets 5, rather than 4. In this case the SPNE is \((NI, (S, L))\).

\[
\text{SPNE : } \{(I, (L, S)), (NI, (S, L)), (I, (1/2,1/3))\}
\]
Example 4

Two sisters, Alicia and Beatriz want to share two euros.

- First, Alice decides whether to have an equal share or not.
- If she decides the equal share, the game is over and the euro is equally divided.
- If she decides not to have an equal share, they play the matching pennies game. If they match, Alice gets 1.25 euros and Beatriz gets 0.75. If they don’t match, Beatriz gets 1.5 and Alice gets 0.5.

a) Represent the extensive form of the game.
b) Find the set SPNE.
NE in subgame Al.2

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1.25, 0.75</td>
<td>0.5, 1.5</td>
</tr>
<tr>
<td>T</td>
<td>0.5, 1.5</td>
<td>1.25, 0.75</td>
</tr>
</tbody>
</table>

\[
q1.25 + 0.5(1 - q) = 0.5q + 1.25(1 - q) \Rightarrow q = 0.5
\]

\[
0.75p + 1.5(1 - p) = 1.5p + 0.75(1 - p) \Rightarrow p = 0.5
\]

Payoffs in NE (0.875, 1.125)
SPNE

If “equal share”: Alicia gets 1
If “not equal share”: Alicia gets 0.875
Alicia prefers E

\[ \text{SPNE} = ((I, 1/2), 1/2) \]
Example 5

Consider again the game between Alicia and Beatriz, but now, in the first stage both have to simultaneously decide whether they want the equal share (E) or not (NE).

If they both say NE, they play a second stage game of matching pennies as before.

If one says E and the other says NE, the game is over, and the one that said E gets 1.25. Compute the new SPNE.

**Information Sets:** Both Alicia and Beatriz have 2 IS each.

**Subgames:** This game has only a subgame (besides the whole game): the one that starts after both say NE.

**Strategies:** Both players’ strategies are vectors of 2 components.
To compute the SPNE replace the subgame after (NE,NE) with its Nash Equilibrium payoffs, as computed in Example 4.

The resulting game is:

```
   A.1
     /   \
    /     \
  E_A    NE_A
    |      |
   / \\   / \\  
  |   |  |   |  B.1
  /   /   /   /
E_B NE_B E_B NE_B
```

1 1.25 0.75 0.875
1 0.75 1.25 1.125
Find the normal form to compute the Nash Equilibria:

<table>
<thead>
<tr>
<th></th>
<th>$E_B$</th>
<th>$NE_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_A$</td>
<td>1,1</td>
<td>1.25, 0.75</td>
</tr>
<tr>
<td>$NE_A$</td>
<td>0.75, 1.25</td>
<td>0.875, 1.125</td>
</tr>
</tbody>
</table>

As $E_i$ strictly dominates $NE_i$ for $i = A, B$, there is a unique NE: $(E_A, E_B)$.

Then:

$$SPNE = \{(E_A, E_B), (1/2, 1/2)\}$$

Or, using a different notation:

$$ENPS = \{(E_A, 1/2), (E_B, 1/2)\}$$