Masters in Economics-uc3m, Microeconomics II: Final Exam (May 24, 2023)

Solve all three exercises.

Exercise 1. (30 points) A pure exchange economy operates over two dates, today and tomorrow. The state of nature tomorrow is uncertain and can be either sunny (S) or cloudy (C). There is a single perishable good, consumption, and two consumers. The consumers' preferences for consumption today (x), consumption tomorrow if sunny (y_S), and consumption tomorrow if cloudy (y_C) are represented by the utility functions $u_1(x, y_S, y_C) = x(2y_S + y_C)$ and $u_2(x, y_S, y_C) = x(y_S + 2y_C)$, respectively, and both are have the initial endowments $(4, 2, 2)$. There are no contingent markets, but there are spot markets for consumption at each date, as well as a credit market and a market for a security both operating today. The security pays 1 unit of the good in tomorrow if sunny and nothing otherwise. Verify that the security price $q^* = 1$ and interest rate $r^* = -1/2$ lead to a competitive equilibrium, and calculate the corresponding allocation. (Hint. Write the consumers budget constraints for (q^*, r^*). Note that consumers' marginal rates of substitution between y_S and y_C are constant, and while it is 2 for consumer 1, it is 1/2 for consumer 2. Take it as a fact that the ratio of the effective prices of y_S and y_C implied by (q^*, r^*) is in the interval $(1/2, 2)$ and derive the consequences over the consumption of y_S and y_C by consumers 1 and 2. Use these results and the budget constrains to simplify the problem of calculating the consumers' demands of credit and security for (q^*, r^*). Then verify that consumers' decisions clear the markets for credit and security. Finally, calculate CE allocation.)

Exercise 2. The revenue of a risk-neutral Principal is a random variable $X(e)$ taking values $x_1 = 24$ and $x_2 = 8$ with probabilities $p(e)$ and $1-p(e)$, that depend on the effort for an agent. The feasible effort levels are $\{1/2, 1\}$, and the corresponding probabilities are $p(1/2) = 1/4$ and $p(1) = 3/4$. There are two types of agents L and H present in fractions $q \in (0, 1)$ and $1 - q$, respectively. All agents' have the same preferences, which are represented by the Bernoulli utility function $u(w) = \sqrt{2w}$, and the same reservation utility, $\underline{u} = 0$. However, their costs of effort differ: while it is $c_L(e) = 2e$ for type L agents, it is $c_H(e) = 4e$ for agents of type H.

(a) (10 points) Determine the contracts the principal will offer if he *observes* the agent's type and effort is *verifiable.*

(b) (15 points) Determine the contracts the principal will offer if he *observes* the agent's type but effort is not *verifiable.*

(c) (15 points) Now assume that effort is verifiable, but the Principal *does not* observe the agent's type. Identify the Principal's optimal menu of contracts for each value of q.

Exercise 3. Two fishermen, Art (A) and Bob (B), have free access to a lake. The total weekly catch of fish of each fisherman $i \in \{A, B\}$ depends on the number of days he fishes, z_i, and the total number of days the two men fish, $z_A + z_B$, according to the formula

$$\frac{8z_i}{3\sqrt{z_A + z_B}}.$$

Their preferences for fish and leisure are identical and are described by the utility function $u(x, y) = x + y$, where x is weakly fish consumption and y is the number of days of leisure during the week.

(a) (15 points) Calculate how much time each fisherman will allocate to leisure and fishing. (You may assume that equilibrium is symmetric.)

(b) (15 points) Determine the socially optimal number of days the two men should fish, and the set of Pareto optimal allocations.

(c) (20 points, extra credit) Assume that the lake's fishing rights are given to a firm owned in equal shares by Art and Both. The firm hires labor l to fish, and sells in the local market its catch of fish, which (consistently with the formula above) is

$$F(l) = \frac{8l}{3\sqrt{l}} = \frac{8}{3}\sqrt{l}.$$

Art and Bob supply labor and use their labor income and their share of the firms profits to buy fish in the local market (in which they are the only buyers). Identify the competitive equilibrium allocation of the economy.

Solutions

Exercise 1. The budget constraints of consumers 1 and 2 are identical and are binding in equilibrium (because each u_i is increasing in all goods), and they are

$$
\begin{aligned}
x &= 4 + b - qs \\
y_S &= 2 - (1+r)b + s \\
y_C &= 2 - (1+r)b.
\end{aligned}
$$

If the ratio of the effective prices of goods y_S and y_C is in $(1/2, 2)$, then in equilibrium $y_{1C} = y_{2S} = 0$, which implies

$$b_1 = \frac{2}{1+r}$$

and

$$s_2 = (1+r)b_2 - 2.$$

Solving for x_{1C} and y_{1C} as consumer 1's utility as a function of s (her demand of security), we may rewrite her problem as

$$\max_{s \in \mathbb{R}} (4 + \frac{2}{1+r} - qs)\,(2s)\,.$$

Taking derivative and solving the first order condition for an interior solution we get

$$s_1(q,r) = \frac{2r+3}{q(1+r)}.$$

Likewise, we may calculate both consumer 2's consumption today x and consumption tomorrow if cloudy y_C as a function of b and rewrite her problem as

$$\max_{b \in \mathbb{R}} \left(4 + b - q((1+r)b - 2\right)\left(2\left(2 - (1+r)b\right)\right).$$

Taking derivative and solving the first order condition for an interior solution we get

$$b_2(q,r) = \frac{2q(1+r) + 2r + 1}{(1+r)\left(q\left(1+r\right) - 1\right)},$$

and hence

$$s_2(q,r) = \frac{2q(1+r) + 2r + 1}{q\left(1+r\right) - 1} - 2.$$

The CE equilibrium interest rate and security price, (r^*, q^*), solve the system

$$\begin{aligned} \frac{2}{1+r} + \frac{2q(1+r)+2r+1}{(1+r)\left(q\left(1+r\right)-1\right)} &= 0 \\ \frac{2r+3}{q(1+r)} + \frac{2q(1+r)+2r+1}{q\left(1+r\right)-1} - 2 &= 0 \end{aligned}$$

which yields

$$(q^*, r^*) = \left(1, -\frac{1}{2}\right).$$

Substituting, we get

$$b_1(q^*, r^*) = -b_2(q^*, r^*) = 4,$$

and

$$s_1(q^*, r^*) = -s_2(q^*, r^*) = 4.$$

Using the budget constraints we calculate the CE allocation:

Hence the equilibrium allocation is

$$[(x_1^*, y_{1S}^*, y_{1C}^*), (x_2^*, y_{2S}^*, y_{2C}^*)] = [(4, 4, 0), (4, 0, 4)]$$

Exercise 2.

(a) Expected revenues are

$$E[X(\frac{1}{2})] = 24\left(\frac{1}{4}\right) + 8\left(1 - \frac{1}{4}\right) = 12, \text{ and } E[X(1)] = 24\left(\frac{3}{4}\right) + 8\left(1 - \frac{3}{4}\right) = 20$$

Since the Principal is risk-neutral, the optimal wage offer is a fixed wage identified by the agents' participation constraints:

$$\sqrt{2w_L(e)} = 0 + 2e, \text{ and } \sqrt{2w_H(e)} = \underline{0} + 4e.$$

Hence

$$\begin{aligned} \bar{w}_L(\frac{1}{2}) &= \frac{1}{2}, \ \bar{w}_L(1) = 2 \\ \bar{w}_H(\frac{1}{2}) &= 2, \ \bar{w}_H(1) = 8 \end{aligned}$$

Thus, the Principal profits for each type of agent and effort level are

$$E[\Pi_L(\frac{1}{2})] = E[X(\frac{1}{2})] - \bar{w}_L(\frac{1}{2}) = 11.5, \ E[\Pi_L(1)] = E[X(1)] - \bar{w}_L(1) = 18,$$

$$E[\Pi_H(\frac{1}{2})] = E[X(\frac{1}{2})] - \bar{w}_H(\frac{1}{2}) = 10, \ E[\Pi_H(1)] = E[X(1)] - \bar{w}_H(1) = 12,$$

and therefore the optimal contracts are

$$(e_L, w_L) = (1, 2), (e_H, w_H) = (1, 8),$$

and Principal expected profit is

$$E[\Pi] = qE[\Pi_L(1)] + (1-q)\,E[\Pi_H(1)] = 18q + 12\,(1-q) = 12 + 6q.$$

(b) The contracts involving low effort, $(e_L, w_L) = (1/2, 1/2)$, and $(e_H, w_H) = (1/2, 2)$ are incentive compatible, but not so the contracts involving high effort. Incentive constraints force such contracts to involve a random wage $W = (w_1, w_2)$ satisfying the participation constraint

$$\frac{3}{4}\sqrt{2w_1} + \frac{1}{4}\sqrt{2w_2} \geq 0 + c_i(1),$$

and the incentive constraint

$$\frac{3}{4}\sqrt{2w_1} + \frac{1}{4}\sqrt{2w_2} - c_i(1) \geq \frac{1}{4}\sqrt{2w_1} + \frac{3}{4}\sqrt{2w_2} - c_i\left(\frac{1}{2}\right),$$

for $i \in \{H, L\}$. Obviously these constraints are binding, and therefore by solving the systems formed by these constraints we get

$$W_L = \left(\frac{25}{8}, \frac{1}{8}\right), W_H = \left(\frac{25}{2}, \frac{1}{2}\right)$$

Hence

$$\begin{aligned} E[W_L] &= \left(\frac{3}{4}\right)\frac{25}{8} + \left(\frac{1}{4}\right)\frac{1}{8} = \frac{19}{8} \\ E[W_H] &= \left(\frac{3}{4}\right)\frac{25}{2} + \left(\frac{1}{4}\right)\frac{1}{2} = \frac{19}{2}, \end{aligned}$$

and therefore with moral hazard the optimal contract to offer to agents of type L involve high effort, that is it is the contract $(1; W_L)$, and lead to an expected profit

$$E[\tilde{\Pi}_L] = E[X(1)] - E[W_L] = 20 - \frac{19}{8} = \frac{141}{8},$$

but the optimal contract to offer to agents of type H involve low effort, that is, $(e_H, w_H) = (1/2, 2)$, and lead to an expected profit

$$E[\tilde{\Pi}_H] = E[X(\frac{1}{2})] - w_H(\frac{1}{2}) = 12 - 2 = 10.$$

Thus, with moral hazard the Principal expected profit is

$$E[\tilde{\Pi}] = qE[\tilde{\Pi}_L] + (1-q)\, E[\tilde{\Pi}_H] = \frac{141}{8}q + 10\,(1-q) = 10 + \frac{61}{8}q.$$

and the reduction of profit due to moral hazard is

$$E[\Pi] - E[\tilde{\Pi}] = 12 + 6q - \left(10 + \frac{61}{8}q\right) = 2 - \frac{13}{8}q > 0$$

(c) The Principal may offer the single contract $(1,2)$*, which only agents of type* L *accept, leading to the an expected profit of*

$$q\left(E[X(1)]-2\right)=18q.$$

Clearly this contract will tend to be optimal when the fraction of low cost agent is near one.

Alternatively, the Principal may design an incentive compatible menu of contracts, $\{(e_L,w_L),(e_H,w_H$
satisfying the participation constraints,

$$\begin{aligned}\sqrt{2w_L} &\geq 0+c_L\left(e_L\right)\\ \sqrt{2w_H} &\geq 0+c_H\left(e_H\right)\end{aligned}$$

and the incentive constraints

$$\begin{aligned}\sqrt{2w_L}-c_L\left(e_L\right) &\geq \sqrt{2w_H}-c_L\left(e_H\right)\\ \sqrt{2w_H}-c_H\left(e_H\right) &\geq \sqrt{2w_L}-c_H\left(e_L\right).\end{aligned}$$

As shown in class the participation constraint for type L *is implied by the participation constraint for type* H *and the incentive constraint for type* L*. Moreover, the participation constraint for type* L *is binding, that is, the principal must give type* L *the rents he can capture by choose the contract* (e_H,w_H)*,*

$$\sqrt{2w_L}=c_L\left(e_L\right)+\left(\sqrt{2w_H}-c_L\left(e_H\right)\right)$$

(The arguments discussed in class apply to this discrete setting.)

A menu with different levels of effort for high a low cost agents must involve low effort for the type H*, i.e., the contract* $(e_H,w_H)=(1/2,2)$*, and high effort for the type* L*, i.e., the contract* $(e_L,w_L)=(1,w_L)$*, where*

$$\sqrt{2w_L}\geq c_L\left(1\right)+\left(\sqrt{2\left(2\right)}-c_L\left(\frac{1}{2}\right)\right))=2+\left(2-\frac{1}{2}\right)=\frac{7}{2}.$$

Hence $w_L=49/8$*. This menu leads to profits*

$$q\left(E[X(1)]-\frac{49}{8}\right)+(1-q)\left(E[X(\frac{1}{2})]-2\right)=q\left(20-\frac{49}{8}\right)+(1-q)\left(12-2\right)=10+\frac{31}{8}q.$$

Alternatively, a "menu" involving the same effort for both types must pay the same wages as well to be incentive compatible; that is, the principal may offer the contract $(e,w)=(1,8)$*, leading to profits*

$$E[X(1)]-8=12,$$

or the contract $(e, w) = (1/2, 2)$, *leading to profits*

$$E[X(\frac{1}{2})] - 2 = 10.$$

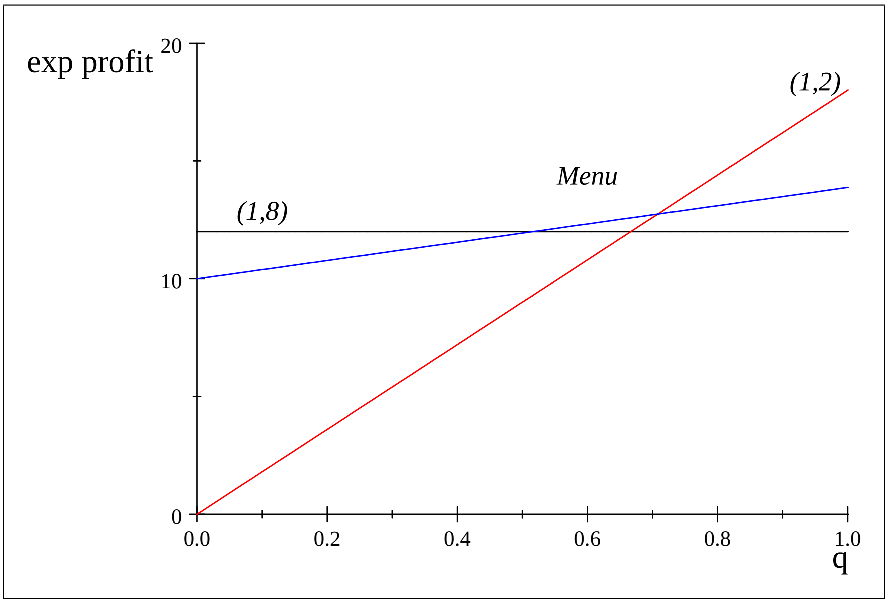
Since

$$10 + \frac{31}{8}q \lesseqqgtr 12 \Leftrightarrow q \lesseqqgtr \frac{16}{31}$$

and

$$10 + \frac{31}{8}q \gtreqqless 18q \Leftrightarrow q \lesseqqgtr \frac{80}{113},$$

then the optimal contract offers are the contract $(1, 2)$ *for* $q \in [0, 16/31]$, *the menu* $\{(1/2, 2), (1, 49/8)\}$ *if* $q \in [16/31, 80/113]$, *and the contract* $(1, 8)$ *if* $q \in [80/113, 1]$ *– see the graph below.*



Exercise 3.

(a) In order to choose z_A Art solves the problem

$$\max_{z_A\in[0,7]} \tfrac{8z_A}{3\sqrt{z_A+z_B}} + 7 - z_B.$$

The F.O.C. for an interior solution is

$$\frac{4z_A + 8z_B - 3\left(z_A+z_B\right)^{\frac{3}{2}}}{3\left(z_A+z_B\right)^{\frac{3}{2}}} = 0.$$

Hence equilibrium is $z_A^* = z_B^* = z$, where z solves the equation

$$4z + 8z - 3\left(z+z\right)^{\frac{3}{2}} = (12 - 6\sqrt{2z})z = 0.$$

Thus, each men fishes $z = 2$ days a week for a total catch of fish equal to $8/3$, and enjoys 5 days of leisure for a total utility equal to $8/3 + 5 = 23/3 = 7.66$.

(*This allocation of time is not Pareto optimal: for example, if both men were to reduce their fishing to just one day, then each men total catch of fish would be also $8/(3\sqrt{2})$, and their days of leisure would be 6, for a total utility of $8/(3\sqrt{2}) + 6 = 7.89$, which makes both men better off.)*

(b) Let us identify the total number of days both men fish, $z \in [0, 14]$, that maximizes social welfare (i.e., the sum of both men utilities). This problem is

$$\max_{z\in[0,14]} \frac{8z}{3\sqrt{z}} + 14 - z.$$

Solving the F.O.C. for an interior solution we get $z^* = 16/9 \simeq 1.78$ days, for a total catch of fish of $32/9 \simeq 3.56$ and a number of days left for leisure activities of $14 - 16/9 = 12.22$. Hence, the set of Pareto optimal allocations is

$$P = \{[(x_1, y_1), (x_2, y_2)] = [(x, y), (32/9 - x, 14 - 16/9 - y),\ x \in [0, 32/9],\ y \in [0, 7]\}$$

(c) (There are many ways to argue the result that I identify below, including the more direct one consisting of calculating it.)

Clearly, the competitive equilibrium allocation is Pareto optimal. Hence, the firm uses the labor $z^* = 16/9$, and supplies $32/9$ units of fish. Also, it is well known that the CE allocation satisfies a property known as "equal treatment," that is, identical agents are treated identically. Therefore both agents enjoy the same number of days of leisure and the same fish consumption, that is, the CE allocation is

$$[(x_1^*, y_1^*), (x_2^*, y_2^*)] = [(16/9, 7 - 8/9), (16/9, 7 - 8/9)].$$