

**CHAPTER 3: Partial derivatives and differentiation**

3-1. Find  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  for the following functions:

- (a)  $f(x, y) = x \cos x \sin y$ .
- (b)  $f(x, y) = e^{xy^2}$ .
- (c)  $f(x, y) = (x^2 + y^2) \ln(x^2 + y^2)$ .

**Solution:**

(a) The partial derivatives of the function  $f(x, y) = x(\cos x)(\sin y)$  are

$$\frac{\partial f(x, y)}{\partial x} = \cos x \sin y - x \sin x \sin y, \quad \frac{\partial f(x, y)}{\partial y} = x \cos x \cos y$$

(b) The partial derivatives of the function  $f(x, y) = e^{xy^2}$  are

$$\frac{\partial f(x, y)}{\partial x} = y^2 e^{xy^2}, \quad \frac{\partial f(x, y)}{\partial y} = 2xy e^{xy^2}$$

(c) The partial derivatives of the function  $f(x, y) = (x^2 + y^2) \ln(x^2 + y^2)$  are

$$\frac{\partial f(x, y)}{\partial x} = 2x \ln(x^2 + y^2) + 2x, \quad \frac{\partial f(x, y)}{\partial y} = 2y \ln(x^2 + y^2) + 2y$$

3-2. Determine the marginal-products for the following Cobb-Douglas production function.

$$F(x, y, z) = 12x^{1/2}y^{1/3}z^{1/4}$$

**Solution:** We compute the partial derivatives with respect to each factor,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 6x^{-1/2}y^{1/3}z^{1/4} \\ \frac{\partial F}{\partial y} &= 4x^{1/2}y^{-2/3}z^{1/4} \\ \frac{\partial F}{\partial z} &= 3x^{1/2}y^{1/3}z^{-3/4} \end{aligned}$$

3-3. Find the gradient of the following functions at the given point  $p$

- (a)  $f(x, y) = (a^2 - x^2 - y^2)^{1/2}$  at  $p = (a/2, a/2)$ .
- (b)  $g(x, y) = \ln(1 + xy)^{1/2}$  at  $p = (1, 1)$ .
- (c)  $h(x, y) = e^y \cos(3x + y)$  at  $p = (2\pi/3, 0)$ .

**Solution:**

(a)  $\nabla(a^2 - x^2 - y^2)^{1/2} = \left( \frac{-x}{\sqrt{(a^2 - x^2 - y^2)}}, \frac{-y}{\sqrt{(a^2 - x^2 - y^2)}} \right)$  so the gradient is

$$\left( \frac{-x}{\sqrt{(a^2 - x^2 - y^2)}}, \frac{-y}{\sqrt{(a^2 - x^2 - y^2)}} \right) \Big|_{x=y=a/2} = \left( \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right)$$

(b)  $\nabla(\ln(1 + xy)^{1/2}) = \frac{1}{2} \left( \frac{y}{1+xy}, \frac{x}{1+xy} \right)$  so the gradient is  $(\frac{1}{4}, \frac{1}{4})$ .

(c)

$$\begin{aligned} &\nabla(e^y \cos(3x + y)) \Big|_{x=2\pi/3, y=0} = \\ &= (-3e^y \sin(3x + y), e^y \cos(3x + y) - e^y \sin(3x + y)) \Big|_{x=2\pi/3, y=0} = (0, 1) \end{aligned}$$

3-4. Consider the function

$$f(x, y) = \begin{cases} 2 \frac{x^2 + y^2}{|x| + |y|} \sin(xy) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Find the partial derivatives of  $f$  at the point  $(0, 0)$ .  
 (b) Prove that  $f$  is continuous on all of  $\mathbb{R}^2$ . Hint: Use (proving it) that for  $(x, y) \neq (0, 0)$  we have that

$$0 \leq \frac{\sqrt{x^2 + y^2}}{|x| + |y|} \leq 1$$

- (c) Is  $f$  differentiable at  $(0, 0)$ ?

**Solution:**

- (a) The partial derivative with respect to  $x$  is

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = 0$$

since  $\sin(0) = 0$ . Likewise,

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = 0$$

so,

$$\nabla f(0, 0) = (0, 0)$$

- (b) Note that for  $(x, y) \neq (0, 0)$ , the function  $f$  is a quotient of continuous functions and, hence is continuous for  $(x, y) \neq (0, 0)$ .

Recall that the function  $f$  is continuous at the point  $(0, 0)$  if

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$$

We prove this next. First, since  $|x|, |y| \geq 0$  then

$$\begin{aligned} x^2 + y^2 &\leq |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2 \end{aligned}$$

Hence,

$$0 \leq \frac{\sqrt{x^2 + y^2}}{|x| + |y|} \leq 1$$

Given  $\varepsilon > 0$ , we take  $\delta = \varepsilon/2$ . If  $0 < \sqrt{x^2 + y^2} < \delta$  then

$$\begin{aligned} |f(x, y)| &= 2 \frac{x^2 + y^2}{|x| + |y|} |\sin(xy)| \quad (\text{since } |\sin(xy)| \leq 1) \\ &\leq 2 \frac{x^2 + y^2}{|x| + |y|} \\ &= 2 \sqrt{x^2 + y^2} \frac{\sqrt{x^2 + y^2}}{|x| + |y|} \quad (\text{by the above observation}) \\ &\leq 2 \sqrt{x^2 + y^2} < 2\delta = \varepsilon \end{aligned}$$

- (c) First, we note that  $f$  is differentiable in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , because the partial derivatives exist and are continuous at every point of  $\mathbb{R}^2 \setminus \{(0, 0)\}$ . The function is differentiable at  $(0, 0)$  if

$$\lim_{(v_1, v_2) \rightarrow (0, 0)} \frac{f(v_1, v_2) - f(0, 0) - \nabla f(0, 0) \cdot (v_1, v_2)}{\sqrt{v_1^2 + v_2^2}} = 0$$

Note that  $f(0, 0) = 0$ ,  $\nabla f(0, 0) \cdot (v_1, v_2) = 0$ . So, let us consider the quotient

$$\frac{f(v_1, v_2)}{\sqrt{v_1^2 + v_2^2}} = 2 \frac{v_1^2 + v_2^2}{(|v_1| + |v_2|) \sqrt{v_1^2 + v_2^2}} \sin(v_1 v_2) = 2 \frac{\sqrt{v_1^2 + v_2^2}}{(|v_1| + |v_2|)} \sin(v_1 v_2)$$

con  $(v_1, v_2) \neq (0, 0)$ . The function  $f$  is differentiable at  $(0, 0)$  if

$$\lim_{(v_1, v_2) \rightarrow (0, 0)} \frac{\sqrt{v_1^2 + v_2^2}}{(|v_1| + |v_2|)} \sin(v_1 v_2) = 0$$

By the observation made in the previous part, we have that

$$0 \leq \frac{\sqrt{v_1^2 + v_2^2}}{(|v_1| + |v_2|)} |\sin(v_1 v_2)| \leq |\sin(v_1 v_2)|$$

and since  $\sin(v_1 v_2)$  is continuous,

$$\lim_{(v_1, v_2) \rightarrow (0, 0)} \sin(v_1 v_2) = 0$$

Hence,

$$\lim_{(v_1, v_2) \rightarrow (0, 0)} \frac{\sqrt{v_1^2 + v_2^2}}{(|v_1| + |v_2|)} \sin(v_1 v_2) = 0$$

and  $f$  is differentiable at  $(0, 0)$ .

3-5. Consider the function

$$f(x, y) = \begin{cases} \frac{x \sin y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Study the continuity of  $f$  in  $\mathbb{R}^2$ .  
 (b) Compute the partial derivatives of  $f$  at the point  $(0, 0)$ .  
 (c) At which points is  $f$  differentiable?

**Solution:**

- (a) Note that

$$\lim_{t \rightarrow 0} f(t, 0) = \lim_{t \rightarrow 0} \frac{0}{t^2} = 0$$

On the other hand since,

$$\lim_{t \rightarrow 0} f(t, t) = \lim_{t \rightarrow 0} \frac{t \sin t}{2t^2} = \lim_{t \rightarrow 0} \frac{\sin t}{2t} = \frac{1}{2}$$

do not coincide, the limit

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$$

does not exist, and hence  $f$  is not continuous at  $(0, 0)$ . The function is continuous at  $\mathbb{R}^2 \setminus \{(0, 0)\}$  since it is a quotient of continuous functions and the denominator does not vanish there.

- (b) The partial derivative with respect to  $x$  is

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t^3} = 0$$

since  $\sin(0) = 0$ . Similarly,

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t^3} = 0$$

- (c) First, we note that  $f$  is differentiable on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , since the partial derivatives exist and are continuous there. The function is not differentiable at  $(0, 0)$  because it is not continuous at that point.

3-6. Consider the function

$$f(x, y) = \begin{cases} 2 \frac{x^3 y}{x^2 + 2y^2} \cos(xy) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- (a) Find the partial derivatives of  $f$  at the point  $(0, 0)$ .  
 (b) Prove that  $f$  is continuous on all of  $\mathbb{R}^2$ . Hint: Note that for  $(x, y) \neq (0, 0)$  we have that

$$\frac{1}{x^2 + 2y^2} \leq \frac{1}{x^2 + y^2}$$

- (c) Is  $f$  differentiable at  $(0, 0)$ ?

**Solution:**

- (a) The partial derivatives are

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \rightarrow 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{t^3} = 0$$

and

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{t \rightarrow 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{0}{2t^3} = 0$$

- (b) The function is continuous on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , since it is a quotient of continuous functions and the denominator does not vanish. Let us study the continuity at the point  $(0, 0)$ . Let  $\varepsilon > 0$ . Take  $\delta = \sqrt{\varepsilon/2}$ . If  $0 < \sqrt{x^2 + y^2} < \delta$  then,

$$\begin{aligned} \left| 2 \frac{x^3 y}{x^2 + 2y^2} \cos(xy) \right| &= 2 \frac{x^2 |x| |y|}{x^2 + 2y^2} |\cos(xy)| \\ &\leq 2|x||y| \quad (\text{since } x^2 \leq x^2 + 2y^2 \text{ y } |\cos(xy)| \leq 1) \\ &= 2\sqrt{x^2} \sqrt{y^2} \leq 2 \left( \sqrt{x^2 + y^2} \right) \left( \sqrt{x^2 + y^2} \right) < 2\delta^2 = \varepsilon \end{aligned}$$

- (c) The function is differentiable at  $(0, 0)$  if

$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x, y) - f(0, 0) - \nabla f(0, 0) \cdot (x, y)}{\sqrt{x^2 + y^2}} = 0$$

Since,  $f(0, 0) = 0$ ,  $\nabla f(0, 0) \cdot (x, y) = 0$ , the function is differentiable at  $(0, 0)$  if

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 y}{(x^2 + 2y^2) \sqrt{x^2 + y^2}} \cos(xy) = 0$$

Given  $\varepsilon > 0$ , we take  $\delta = \varepsilon$ . If  $0 < \sqrt{x^2 + y^2} < \delta$  then,

$$\begin{aligned} \left| \frac{x^3 y}{(x^2 + 2y^2) \sqrt{x^2 + y^2}} \cos(xy) \right| &\leq \left| \frac{x^3 y}{(x^2 + 2y^2) \sqrt{x^2 + y^2}} \right| \\ &= \frac{x^2 |x| |y|}{(x^2 + 2y^2) \sqrt{x^2 + y^2}} \\ &\leq \frac{|x| |y|}{\sqrt{x^2 + y^2}} = \frac{\sqrt{x^2} \sqrt{y^2}}{\sqrt{x^2 + y^2}} \\ &\leq \sqrt{y^2} \leq \sqrt{x^2 + y^2} < \delta = \varepsilon \end{aligned}$$

so  $f$  is differentiable at  $(0, 0)$ .

3-7. Compute the derivatives of the following functions at the given point  $p$  along the vector  $v$

- (a)  $f(x, y) = x + 2xy - 3y^2$ ,  $p = (1, 2)$ ,  $v = (3, 4)$ .  
 (b)  $g(x, y) = e^{xy} + y \tan^{-1} x$ ,  $p = (1, 1)$ ,  $v = (1, -1)$ .  
 (c)  $h(x, y) = (x^2 + y^2)^{1/2}$ ,  $p = (0, 5)$ ,  $v = (1, -1)$ .

**Solution:**

- (a)  $\nabla(x + 2xy - 3y^2)|_{x=1, y=2} = (1 + 2y, 2x - 6y)|_{x=1, y=2} = (5, -10)$ . So, the derivative along the vector  $(3, 4)$  is

$$(5, -10) \cdot (3, 4) = -25$$

- (b)  $\nabla(e^{xy} + y \arctan x)|_{x=1, y=1} = \left( ye^{xy} + \frac{y}{1+x^2}, xe^{xy} + \arctan x \right)|_{x=1, y=1} = (e + \frac{1}{2}, e + \arctan 1) = (e + \frac{1}{2}, e + \frac{\pi}{4})$ . So, the derivative along the vector  $(1, -1)$  is

$$(e + \frac{1}{2}, e + \frac{\pi}{4}) \cdot (1, -1) = \frac{1}{2} - \frac{\pi}{4}$$

- (c)  $\nabla((x^2 + y^2)^{1/2})|_{x=0, y=5} = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)|_{x=0, y=5} = (0, 1)$ . So, the derivative along the vector  $(1, -1)$  is

$$(0, 1) \cdot (1, -1) = -1$$

3-8. Let  $B(x, y) = 10x - x^2 - \frac{1}{2}xy + 5y$  be the profits of a firm. Last year the company sold  $x = 4$  units of good 1 and  $y = 2$  units of good 2. This year, the company can change slightly the amounts of the goods  $x$  and  $y$  it sells. If it wishes to increase its profit as much as possible, what should  $\frac{\Delta x}{\Delta y}$  be?

**Solution:**

$$\nabla(10x - x^2 - \frac{xy}{2} + 5y)|_{x=4, y=2} = \left( 10 - 2x - \frac{y}{2}, -\frac{x}{2} + 5 \right)|_{x=4, y=2} = (1, 3)$$

Since the gradient points in the direction of maximum growth of the function, if there is an increase  $(\Delta x, \Delta y)$ , for the function to increase the most, we must have that  $(\Delta x, \Delta y) = k(1, 3)$ . From here we obtain that  $\Delta x = k$   $\Delta y = 3k$ . Hence,  $\Delta x/\Delta y = 1/3$ .

3-9. Knowing that  $\frac{\partial f}{\partial x}(2, 3) = 7$  and  $D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}f(2, 3) = 3\sqrt{5}$ , find  $\frac{\partial f}{\partial y}(2, 3)$  and  $D_v f(2, 3)$  with  $v = (\frac{3}{5}, \frac{4}{5})$ .

**Solution:** We know that

$$D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}f(2, 3) = \left( \frac{\partial f}{\partial x}(2, 3), \frac{\partial f}{\partial y}(2, 3) \right) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) = 3\sqrt{5}$$

and also

$$\frac{\partial f}{\partial x}(2, 3) = 7$$

Letting

$$z = \frac{\partial f}{\partial y}(2, 3)$$

we have that

$$(7, z) \cdot \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right) = 3\sqrt{5}$$

But this is equivalent to

$$\frac{7\sqrt{5}}{5} + \frac{2z\sqrt{5}}{5} = 3\sqrt{5}$$

And, therefore

$$\frac{\partial f}{\partial y}(2, 3) = 4$$

We may compute now

$$D_{(\frac{3}{5}, \frac{4}{5})}f(2, 3) = \left( \frac{\partial f}{\partial x}(2, 3), \frac{\partial f}{\partial y}(2, 3) \right) \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = (7, 4) \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = \frac{37}{5}$$

3-10. Find the derivative of  $f(x, y, z) = xy^2 + z^2y$ , along the vector  $v = (1, -1, 2)$  at the point  $(1, 1, 0)$ . Determine the direction which maximizes (resp. minimizes) the directional derivative at the point  $(1, 1, 0)$ . What are the largest and smallest values of the directional derivative at that point?

**Solution:** The gradient of the function  $f(x, y, z) = xy^2 + z^2y$  at the point  $(1, 1, 0)$  is

$$\nabla f(1, 1, 0) = \nabla(xy^2 + z^2y)|_{x=1, y=1, z=0} = (y^2, 2xy + z^2, 2zy)|_{x=1, y=1, z=0} = (1, 2, 0)$$

The derivative along  $v$  is

$$D_v f(1, 1, 0) = \nabla f(1, 1, 0) \cdot v = (1, 2, 0) \cdot (1, -1, 2) = -1$$

The direction which maximizes the directional derivative is

$$\frac{\nabla f(1, 1, 0)}{\|\nabla f(1, 1, 0)\|} = \frac{1}{\sqrt{5}}(1, 2, 0)$$

and the maximum value of the directional derivative is  $\|\nabla f(1, 1, 0)\| = \sqrt{5}$ .

Likewise, the direction which minimizes the directional derivative is

$$-\frac{\nabla f(1, 1, 0)}{\|\nabla f(1, 1, 0)\|} = \frac{1}{\sqrt{5}}(-1, -2, 0)$$

and the minimum value of the directional derivative is  $-\|\nabla f(1, 1, 0)\| = -\sqrt{5}$ .

3-11. Consider the function  $f(x, y) = x^2 + y^2 + 1$   $g(x, y) = (x + y, ay)$ . Determine:

- The value of  $a$  for which the function  $f \circ g$  grows fastest in the direction of the vector  $v = (5, 7)$  at the point  $p = (1, 1)$ .
- The equations of the tangent and normal lines to the curve  $xy^2 - 2x^2 + y + 5x = 6$  at the point  $(4, 2)$ .

**Solution:** Consider the functions  $f(x, y) = x^2 + y^2 + 1$   $g(x, y) = (x + y, ay)$

- (a) Their composition is  $f(g(x, y)) = f(x + y, ay) = (x + y)^2 + a^2y^2 + 1$  and the gradient at the point  $(1, 1)$  is

$$\nabla(f(g(1, 1))) = (2x + 2y, 2x + 2y + 2a^2y)|_{x=1, y=1} = (4, 4 + 2a^2)$$

If we want that direction of the vector  $v = (5, 7)$  is the direction of maximum growth of  $f(g(x, y))$  at the point  $(1, 1)$ , we must have that  $v$  and  $\nabla(f(g(x, y)))(1, 1)$  are parallel. That is,

$$\frac{4 + 2a^2}{4} = \frac{7}{5}$$

whose solution is

$$a = \pm \frac{2}{\sqrt{5}}$$

- (b) Note first that the point  $(4, 2)$  satisfies the equation  $xy^2 - 2x^2 + y + 5x = 6$ . Now, the gradient of the function  $g(x, y) = xy^2 - 2x^2 + y + 5x = 6$  at the point  $(4, 2)$  is

$$\nabla g(4, 2) = (y^2 - 4x + 5, 2xy + 1)|_{\substack{x=4 \\ y=2}} = (-7, 17)$$

Thus, the equation of the tangent line is

$$(-7, 17) \cdot (x - 4, y - 2) = 0$$

and the parametric equations of the normal line are

$$(x(t), y(t)) = (4, 2) + t(-7, 17)$$

3-12. Find the Jacobian matrix of  $F$  in the following cases.

- (a)  $F(x, y, z) = (xyz, x^2z)$   
 (b)  $F(x, y) = (e^{xy}, \ln x)$   
 (c)  $F(x, y, z) = (\sin xyz, xz)$

**Solution:**

- (a) The Jacobian matrix of  $F$  is

$$D F(x, y, z) = \begin{pmatrix} yz & xz & xy \\ 2xz & 0 & x^2 \end{pmatrix}$$

- (b) The Jacobian matrix of

$$D F(x, y, z) = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 1/x & 0 \end{pmatrix}$$

- (c) The Jacobian matrix of

$$D F(x, y, z) = \begin{pmatrix} yz \sin xyz & xz \sin xyz & xy \sin xyz \\ z & 0 & x \end{pmatrix}$$

3-13. Using the chain rule compute the derivatives

$$\frac{\partial z}{\partial r} \quad \frac{\partial z}{\partial \theta}$$

in the following cases.

- (a)  $z = x^2 - 2xy + y^2$ ,  $x = r + \theta$ ,  $y = r - \theta$   
 (b)  $z = \sqrt{25 - 5x^2 - 5y^2}$ ,  $x = r \cos \theta$ ,  $y = r \sin \theta$

**Solution:**

- (a)

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\ &= 2x - 2y - 2x + 2y = 0 \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} \\ &= 2x - 2y - (-2x + 2y) = 4(x - y) = 8\theta \end{aligned}$$

(b)

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\
&= -\frac{10x}{2\sqrt{25-5x^2-5y^2}} \cos \theta - \frac{10y}{2\sqrt{25-5x^2-5y^2}} \sin \theta \\
&= -\frac{5r \cos^2 \theta}{\sqrt{25-5r^2}} - \frac{5r \sin^2 \theta}{\sqrt{25-5r^2}} = -\frac{5r}{\sqrt{25-5r^2}} \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} \\
&= -\frac{10x}{\sqrt{25-5x^2-5y^2}} + \frac{10y}{\sqrt{25-5x^2-5y^2}} \\
&= -\frac{10x}{2\sqrt{25-5x^2-5y^2}}(-r \sin \theta) - \frac{10y}{2\sqrt{25-5x^2-5y^2}}(r \cos \theta) \\
&= \frac{5r^2 \cos \theta \sin \theta}{\sqrt{25-5r^2}} - \frac{5r^2 \cos \theta \sin \theta}{\sqrt{25-5r^2}} = 0
\end{aligned}$$

3-14. Using the capital  $K$  at time  $t$  generates an instant profit of

$$B(t) = 5(1+t)^{1/2}K$$

Suppose that capital evolves in time according to the equation  $K(t) = 120e^{t/4}$ . Determine the rate of change of  $B$ .

**Solution:**

Since

$$\frac{dK}{dt} = 30e^{t/4}$$

we see that

$$\begin{aligned}
\frac{d}{dt}B &= \frac{5}{2}(1+t)^{-1/2}K + 5(1+t)^{1/2}\frac{dK}{dt} \\
&= 300(1+t)^{-1/2}e^{t/4} + 150(1+t)^{1/2}e^{t/4}
\end{aligned}$$

3-15. Verify the chain rule for the function  $h = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$  with  $x = e^t$ ,  $y = e^{t^2}$  and  $z = e^{t^3}$ .

3-16. Verify the chain rule for the composition  $f \circ c$  in the following cases.

(a)  $f(x, y) = xy$ ,  $c(t) = (e^t, \cos t)$ .

(b)  $f(x, y) = e^{xy}$ ,  $c(t) = (3t^2, t^3)$ .

**Solution:**

(a) The functions are  $f(x, y) = xy$  and  $c(t) = (x(t), y(t)) = (e^t, \cos t)$ . Therefore,  $f(x(t), y(t)) = f(e^t, \cos t) = e^t \cos t$  and

$$\frac{d}{dt}f(x(t), y(t)) = e^t \cos t - e^t \sin t$$

Now, we compute

$$\nabla f(c(t)) \cdot \frac{dc}{dt}$$

On the one hand,

$$\nabla f(x, y) = (y, x)$$

and

$$\frac{dc}{dt} = (e^t, -\sin t)$$

Therefore,

$$\nabla f(c(t)) \cdot \frac{dc}{dt} = y(t)e^t - x(t) \sin t$$

which coincides with the computation above.

- (b) The functions are  $f(x, y) = e^{xy}$  and  $c(t) = (x(t), y(t)) = (3t^2, t^3)$ . Therefore,  $f(x(t), y(t)) = f(3t^2, t^3) = e^{3t^5}$  and

$$\frac{d}{dt}f(x(t), y(t)) = 15t^4e^{3t^5}$$

Now, we compute

$$\nabla f(c(t)) \cdot \frac{dc}{dt}$$

On the one hand,

$$\nabla f(x, y) = (ye^{xy}, xe^{xy})$$

and

$$\frac{dc}{dt} = (6t, 3t^2)$$

Therefore,

$$\nabla f(c(t)) \cdot \frac{dc}{dt} = (6ye^{xy}t + 3xe^{xy}t^2)|_{x=3t^2, y=t^3} = 15t^4e^{3t^5}$$

3-17. Write the chain rule  $h'(x)$  in the following cases.

- (a)  $h(x) = f(x, u(x, a))$ , where  $a \in \mathbb{R}$  is a parameter.  
 (b)  $h(x) = f(x, u(x), v(x))$ .

**Solution:**

(a)

$$h'(x) = \frac{\partial f(x, u(x, a))}{\partial x} + \frac{\partial f(x, u(x, a))}{\partial y} \frac{\partial u(x, a)}{\partial x}$$

(b)

$$h'(x) = \frac{\partial f(x, u(x), v(x))}{\partial x} + \frac{\partial f(x, u(x), v(x))}{\partial y} u'(x) + \frac{\partial f(x, u(x), v(x))}{\partial z} v'(x)$$

3-18. Determine the points at which the tangent plane to the surface  $z = e^{(x-1)^2+y^2}$  is horizontal. Determine the equation of the tangent plane at those points.

**Solution:** Consider the function of 3 variables

$$g(x, y, z) = e^{(x-1)^2+y^2} - z$$

We are asked to compute the tangent plane to the level surface

$$A = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$$

at the point  $(x, y, z)$  where this tangent plane is horizontal. At that point we must have that

$$\nabla g(x, y, z) = (0, 0, -1)$$

Since,

$$\nabla g(x, y, z) = (2(x-1)e^{(x-1)^2+y^2}, 2ye^{(x-1)^2+y^2}, -1)$$

we must have that  $x = 1, y = 0$ . The  $z$  coordinate is

$$z = e^{(x-1)^2+y^2} \Big|_{x=1, y=0} = 1$$

And the tangent is horizontal at the point  $(1, 0, 1)$ . The equation of the tangent plane is

$$z = 1$$

3-19. Consider the function  $f(x, y) = (xe^y)^3$ .

- (a) Compute the equation of the tangent plane to the graph of  $f(x, y)$  at the point  $(2, 0)$ .  
 (b) Using the equation of the tangent plane, find an approximation to  $(1, 999e^{0.002})^3$ .

**Solution:**



- (a) We are asked to compute the tangent plane to the graph of  $f$  at the point  $(2, 0, f(2, 0)) = (2, 0, 8)$ . Consider the function of 3 variables

$$g(x, y, z) = x^3 e^{3y} - z$$

The graph of  $f$  is the level surface of  $g$ ,

$$A = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$$

Therefore, it is enough to compute the tangent plane to the level surface  $A$  at the point  $(2, 0, 8)$ . Since,

$$\nabla g(2, 0, 8) = (3x^2 e^y, 3x^3 e^y, -1)|_{x=2, y=0, z=8} = (12, 24, -1)$$

the equation of the tangent plane is

$$(12, 24, -1) \cdot (x - 2, y, z - 8) = 0$$

that is,

$$12x + 24y - z = 16$$

- (b) We are asked to estimate the function  $f(x, y) = x^3 e^{3y}$  at the point  $(1'999, 0'002)$ . Since,  $f$  is differentiable and that point is very close to  $(2, 0)$ , we use Taylor's first order approximation around the point  $(2, 0)$ ,

$$z = 12x + 24y - 16$$

and we obtain that

$$f(1'999, 0'002) \approx (12x + 24y - 16)|_{x=1'999, y=0'002} = 8'036$$

3-20. Compute the tangent plane and normal line to the following level surfaces.

- (a)  $x^2 + 2xy + 2y^2 - z = 0$  at the point  $(1, 1, 5)$ .  
 (b)  $x^2 + y^2 - z = 0$  at the point  $(1, 2, 5)$ .  
 (c)  $(y - x^2)(y - 2x^2) - z = 0$  at the point  $(1, 3, 2)$ .

**Solution:**

- (a) We compute the gradient

$$\nabla(x^2 + 2xy + 2y^2 - z)|_{(x,y,z)=(1,1,5)} = (2x + 2y, 2x + 4y, -1)|_{(x,y,z)=(1,1,5)} = (4, 6, -1)$$

Thus, the equation of the tangent plane is

$$(4, 6, -1) \cdot (x - 1, y - 1, z - 5) = 0$$

that is,

$$4x + 6y - z = 5$$

- (b) We compute the gradient

$$\nabla(x^2 + y^2 - z)|_{(x,y,z)=(1,2,5)} = (2x, 2y, -1)|_{(x,y,z)=(1,2,5)} = (2, 4, -1)$$

Thus, the equation of the tangent plane is

$$(2, 4, -1) \cdot (x - 1, y - 2, z - 5) = 0$$

that is,

$$2x + 4y - z = 5$$

- (c) We compute the gradient

$$\begin{aligned} \nabla((y - x^2)(y - 2x^2) - z)|_{(x,y,z)=(1,3,2)} &= (-2x(y - 2x^2) - 4x(y - x^2), 2y - 3x^2, -1)|_{(x,y,z)=(1,3,2)} \\ &= (-10, 3, -1) \end{aligned}$$

Thus, the equation of the tangent plane is

$$(-10, 3, -1) \cdot (x - 1, y - 3, z - 2) = 0$$

that is,

$$10x - 3y + z = 3$$

3-21. Let  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be two functions with continuous partial derivatives on  $\mathbb{R}^2$ .

- (a) Show that if

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial g}{\partial x}(x, y)$$

at every point  $(x, y) \in \mathbb{R}^2$ , then  $f - g$  depends only on  $y$ .

(b) Show that if

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial g}{\partial y}(x, y)$$

at every point  $(x, y) \in \mathbb{R}^2$ , then  $f - g$  depends only on  $x$ .

(c) Show that if  $\nabla(f - g)(x, y) = (0, 0)$  at every point  $(x, y) \in \mathbb{R}^2$ , then  $f - g$  is constant on  $\mathbb{R}^2$ .

(d) Find a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\frac{\partial f}{\partial y}(x, y) = yx^2 + x + 2y, \quad \frac{\partial f}{\partial x}(x, y) = y^2x + y, \quad f(0, 0) = 1$$

Are there any other functions satisfying those equations?

**Solution:** The functions  $f$  and  $g$  are of class  $C^1$ .

(a) Let  $(a, b), (x, b) \in \mathbb{R}^2$ . Let  $h(x, y) = f(x, y) - g(x, y)$ . By the Mean Value Theorem,

$$h(x, b) - h(a, b) = \nabla h(c) \cdot (x - a, 0)$$

for some point  $c = (c_1, c_2) = t(x, b) + (1 - t)(a, b) = (tx + (1 - t)a, b)$  with  $0 < t < 1$ . Since,

$$\frac{\partial h}{\partial x}(c) = 0$$

we have that  $h(x, b) - h(a, b) = 0$ . That is,  $h(x, b) = h(a, b)$  for every  $x \in \mathbb{R}$  and the function  $h$  does not depend on  $y$ .

(b) Is very similar to the previous case.

(c) At each point of  $\mathbb{R}^2$  we have that

$$\frac{\partial(f - g)}{\partial x} = \frac{\partial(f - g)}{\partial y} = 0$$

so  $f - g$  does not depend neither on  $x$  nor on  $y$ .

(d) We know that  $\frac{\partial f}{\partial y}(x, y) = yx^2 + x + 2y$ . Integrating with respect to  $y$ ,

$$f(x, y) = \int (yx^2 + x + 2y)dy = \frac{1}{2}y^2x^2 + xy + y^2 + C(x)$$

where  $C(x)$  is a function that depends only on  $x$ . The other condition is  $\frac{\partial f}{\partial x}(x, y) = y^2x + y$ . We try this with the function that we have obtained,

$$\frac{\partial}{\partial x} \left( \frac{1}{2}y^2x^2 + xy + y^2 + C(x) \right) = y^2x + y + C'(x)$$

so,  $C'(x) = 0$  and  $C(x) = c$ , a constant. To find  $c$  we use the condition  $f(0, 0) = 1$ . Thus,

$$f(x, y) = \frac{1}{2}y^2x^2 + xy + y^2 + c$$

$$f(0, 0) = c \text{ and } f(0, 0) = 1. \text{ Hence } c = 1$$

The function  $f(x, y) = \frac{1}{2}y^2x^2 + xy + y^2 + 1$  satisfies the above conditions. If there were another function  $g$  of class  $C^1$  satisfying the same conditions, we would have that  $\nabla(f - g)(x, y) = (0, 0)$  at every point  $(x, y) \in \mathbb{R}^2$ . By part (c) there is a constant  $A \in \mathbb{R}$  such that

$$(f - g)(x, y) = A$$

for every  $(x, y) \in \mathbb{R}^2$ . But, since  $f(0, 0) = 1 = g(0, 0)$ , we have that  $A = 0$  and the functions coincide.