Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	Total
Points						

Department of Economics Mathematics I Final Exam June 22nd 2023 Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = \dfrac{\sqrt{x^2+5x+4}}{x+2}$. **Then:**

(a) find the domain of the function $f(x)$ and its asymptote at $+\infty$.

(b) now, consider the function $f(x)$ defined on the interval $[0, \infty)$. Find the intervals where $f(x)$ increases and decreases and its range or image. Draw the graph of the function.

(c) let $g(x)$ be a continuous function defined on $[0, \infty)$, increasing in $[0, a]$, decreasing in $[a, \infty)$ and such that satisfies: $g(0) = \lim_{x\to\infty} g(x)$.
Consider $g(x)$, defined on the interval $[b, \infty)$. Discuss the existence of its global extrema depending on the parameters $a, b > 0$.
0.4 points part a); 0.4 points part b); 0.2 points part c)

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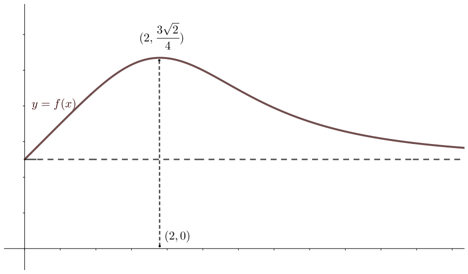
a) First of all, the domain of the function is $\mathbb{R} - (-4, -1)$, since the roots of the polynomial x^2+5x+4 are $-4, -1$ and the values of the polynomial are negative in the interval $(-4, -1)$, so the square root doesn't exist at them. Moreover, we should exclude the point $x = -2$ since, the denominator is zero at this point, but it is already included in the interval $(-4, -1)$.
Secondly, we find the asymptote at $+\infty$, calculating the limit towards ∞ we obtain $\lim_{x\to\infty} \dfrac{\sqrt{x^2+5x+4}}{x+2} =$ (dividing numerator and denominator by x) $= \lim_{x\to\infty} \dfrac{\sqrt{1+5/x+4/x^2}}{1+2/x} = 1$.
Hence, f has a horizontal asymptote $y = 1$ at ∞.
Obviously, since there is a horizontal asymptote then an oblique asymptote doesn't exist on the same side.

b) To find the increasing/decreasing intervals of the function, we study the sign of the derived function of $f(x)$:

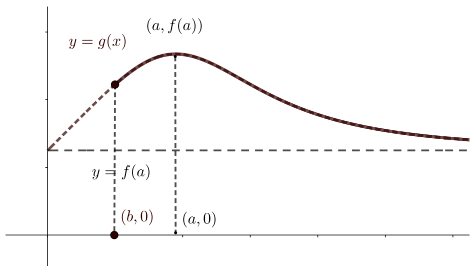
$$f'(x) = \left(\frac{\sqrt{x^2+5x+4}}{x+2}\right)' = \frac{[(2x+5)/2\sqrt{x^2+5x+4})](x+2) - \sqrt{x^2+5x+4}}{(x+2)^2} =$$

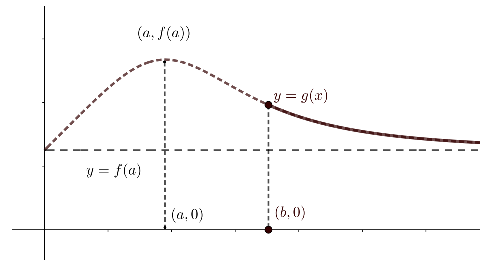
$$= \frac{(2x+5)(x+2) - 2(x^2+5x+4)}{2(x+2)^2\sqrt{x^2+5x+4}} = \frac{(2x^2+9x+10) - (2x^2+10x+8)}{2(x+2)^2\sqrt{x^2+5x+4}} =$$

$= \dfrac{-x+2}{2(x+2)^2\sqrt{x^2+5x+4}}$, since the sign of the denominator is always positive, the sign of derived function depends only on the numerator. We obtain:
i) $f'(x) > 0 \Leftrightarrow x \in (0, 2)$, hence f is increasing in $[0, 2]$.
ii) $f'(x) < 0 \Leftrightarrow x \in (2, \infty)$, hence f is decreasing in $[2, \infty)$.
Furthermore, since $f(x)$ is continuous in its domain, f is icreasing in $[0, 2]$, f is decreasing in $[2, \infty)$ and $\lim_{x\to\infty} f(x) = f(0) = 1$, due to the Intermediate Value Theorem we can deduce that the range of the function on the $[0, \infty)$ will be $[1, f(2)] = [1, \sqrt{18}/4] = [1, \frac{3}{4}\sqrt{2}]$.
The graph of f will have an appearance approximately, similar to the one in the figure underneath.



c) The prevous function $f(x)$ is an example of function that satisfies the conditions of $g(x)$, where $g(0) = 1$ and $a = 2$.
So, clearly, the function $g(x)$ doesn't attain a global minimum since the function approximates the value of $g(0)$ from above, but it is not reached.
Now, for the global maximum we have the following cases:
i) if $b < a$, then the global maximum is attained at the point a.
ii) if $a < b$, then the global maximum is attained at the point b.
iii) if $a = b$, then the global maximum is attained at the point $a = b$.
The following drawings will clarify the situation:





(2) **Given the implicit function $y = f(x)$, defined by the equation $e^{2x+y} + x^2 y = e^2$ in a neighbourhood of the point $x = 0, y = 2$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 0$.

(b) sketch the graph of the function f near the point $x = 0, y = 2$.

(c) use the second-order Taylor Polynomial of $f(x)$ to obtain the approximate values of $f(-0.2)$ and $f(0.1)$.

Using this polynomial, compare the value of $f(0)$ with $\frac{1}{3}f(-0,2) + \frac{2}{3}f(0,1)$.

(*Hint for part (b) and (c):* use $f''(0) < 0$).

0.4 points part a); 0.3 points part b); 0.3 points part c).

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a) First of all, we calculate the first-order derivative of the equation:

$e^{2x+y}(2 + y') + 2xy + x^2 y' = 0$

evaluating at $x = 0,\ y(0) = 2$ we obtain: $y'(0) = f'(0) = -2$.

Then the equation of the tangent line is: $y = P_1(x) = 2 - 2x$, or $2x + y = 2$.

Secondly, we calculate the second-order derivative of the equation:

$e^{2x+y}[(2 + y')^2 + y''] + 2y + 2xy' + 2xy' + x^2 y'' = 0$

evaluating at $x = 0, y(0) = 2, y'(0) = -2$ we obtain: $y''(0) = f''(0) = -\frac{4}{e^2}$.

Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 2 - 2x - \frac{2}{e^2}x^2$.

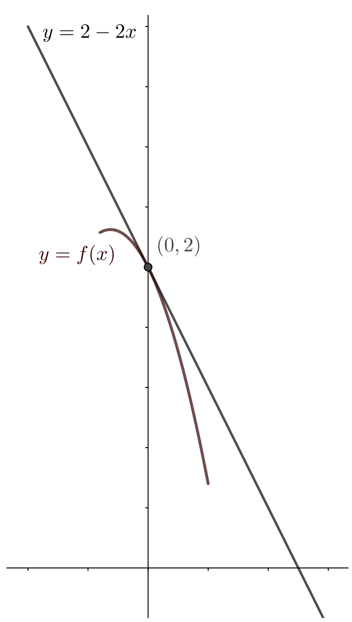
b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 0$ will be as you can see in the figure underneath.

c) Finally, using the second-order Taylor Polynomial we obtain:

$f(-0,2) \approx 2 - 2(-0,2) - \dfrac{2}{e^2}\, 0,04;\ f(0,1) \approx 2 - 2(0,1) - \dfrac{2}{e^2} 0,01 \Longrightarrow$

$\frac{1}{3}f(-0,2) + \frac{2}{3}f(0,1) = 2 - \dfrac{2}{e^2} 0,02 < 2 = f(0) = f(\frac{1}{3}(-0,2) + \frac{2}{3}(0,1))$.

Which is to be expected, since $f(x)$ is a concave function near $x = 0$.



(3) **Let $C(x) = C_0 + 2x + x^2$ be the cost function and $p(x) = 100 - ax$ the inverse demand function of a monopolistic firm, with $a, C_0 > 0$. Then:**

(a) calculate the value of the parameters a, C_0, knowing that the production level to maximize the profit is $x^* = 7$.

(b) calculate the value of the parameters a, C_0, knowing that the production level to minimize the average cost is $x^{**} = 7$.

0.5 points part a); 0.5 points part b).

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a) First of all, we calculate the profit function.
$B(x) = (100 - ax)x - (C_0 + 2x + x^2) = -(a+1)x^2 + 98x - C_0$
Secondly, we calculate the first and second order derivatives of B:
$B'(x) = -2(a+1)x + 98$; $B''(x) = -2(a+1) < 0$
we see that B has a unique critical point at $x^* = \dfrac{98}{2(a+1)}$ and, since B is a concave function, the critical point is the unique global minimizer.

Finally, $x^* = 7 = \dfrac{98}{2(a+1)} \Longrightarrow a = 6$; C_0 can take any value.

b) The average cost function is $\dfrac{C(x)}{x} = x + 2 + \dfrac{C_0}{x}$, its first order derivative: $\left(\dfrac{C(x)}{x}\right)' = 1 - \dfrac{C_0}{x^2} =$ $= 0 \Longleftrightarrow x^2 = C_0$.
Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{2C_0}{x^3} > 0$, the function is convex and the critical point will be the global minimizer.
Then $x^{**} = 7 \Longrightarrow C_0 = 49$; and a can take any value.

(4) **Consider the function** $f(x) = \ln(a + e^x)$, **defined on the interval** $[0, \infty)$, **where** $0 < a$. **Then:**

(a) find the asymptote of $f(x)$ at $+\infty$.

(b) determine the increasing/decreasing and the convexity/concavity intervals of $f(x)$.

(c) sketch the graph of $f(x)$ and its inverse function in the same Cartesian plane. Find the domain of both functions (obviously, it is not necessary to calculate the formula of the f-inverse function nor its derivatives).

(*Hint for part (a):* use that $\ln e^x = x$ and $\ln a - \ln b = \ln \frac{a}{b}$).

0.4 points part a); 0.2 points part b); 0.4 points part c)

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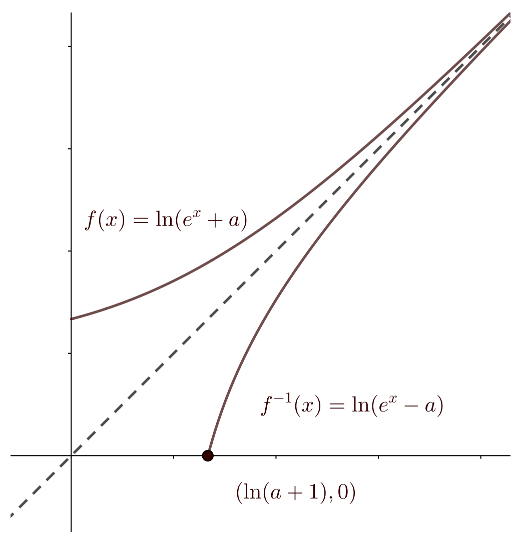
a) $f(x)$ has not vertical asymptotes since the function is continuous in its domain.
Now, for asymptotes at infinity,
$\lim_{x\longrightarrow\infty} \frac{\ln(a+e^x)}{x} = \frac{\infty}{\infty} =$ (L'Hopital)$= \lim_{x\longrightarrow\infty} \frac{e^x}{a+e^x} = \lim_{x\longrightarrow\infty} \frac{1}{(a/e^x)+1} = 1$. Moreover,
$\lim_{x\longrightarrow\infty} \ln(a+e^x) - x = \lim_{x\longrightarrow\infty} \ln(a+e^x) - \ln e^x = \lim_{x\longrightarrow\infty} \ln \frac{a+e^x}{e^x} = \lim_{x\longrightarrow\infty} \ln(\frac{a}{e^x} + 1) =$
$= \ln 1 = 0$, then the oblique asymptote is $y = x$.

b) Obviously, $f\,'(x) = \frac{e^x}{a+e^x} > 0$, so the function is increasing.
Analogously, $f''(x) = \frac{e^x(a+e^x) - e^x e^x}{(a+e^x)^2} = \frac{ae^x}{(a+e^x)^2} > 0$, so $f(x)$ is convex.

c) The graph of f will have an appearance approximately, similar to the one in the figure underneath. Using the symmetric with respect to the first bisector line the graph of its inverse will have an appearance approximately, similar to the one in this figure:



(5) **Given the functions** $f, g : [0, \infty) \longrightarrow \mathbb{R}$, **defined by:** $f(x) = -\ln(1+x)$, $g(x) = 1 - \ln(1+3x)$, **then:**

(a) draw approximately the set A, bounded by the graph of these functions and the straight line $x = 0$. Find, if they exist, the maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

(*Hint for part (a):* remember that $\ln a - \ln b = \ln \frac{a}{b}$).

(*Hint for part (b):* once used Barrow's rule, you **don't** need to simplify the expresion!)

0.6 points part a); 0.4 points part b)

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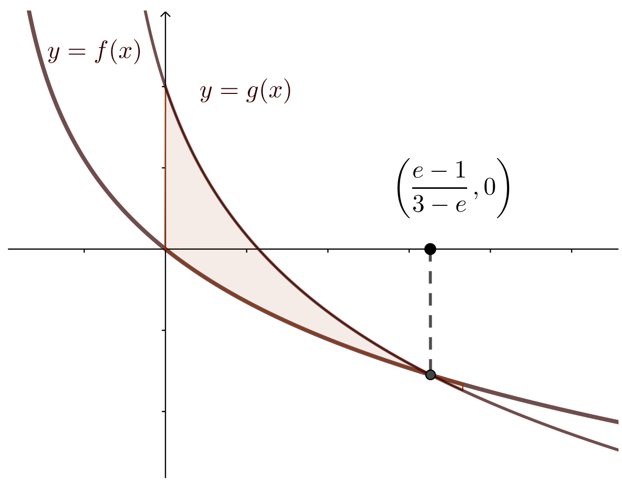
a) First of all we can observe that $f(0) = 0 < 1 = g(0)$. On the other hand both functions intersect each other at:

$-\ln(1+x) = 1 - \ln(1+3x) \iff \ln(\dfrac{1+3x}{1+x}) = 1 \iff \dfrac{1+3x}{1+x} = e \iff 1 + 3x = e + ex \iff$

$\iff (3-e)x = e - 1 \iff x = (e-1)/(3-e) = x^*$.

Moreover, since both functions are decreasing, the draw of A will be approximately like,



Then, Pareto order describes the set properties:

maximum(A) and minimum(A) do not exist.

maximal elements(A) $= \{(x, g(x)) : 0 \leq x \leq x^*\}$.

minimal elements(A) $= \{(x, f(x)) : 0 \leq x \leq x^*\}$.

b) First of all, looking at the position of the graphs we know that:

area(A)$= \int_0^{x^*} (1 - \ln(1+3x) + \ln(1+x))dx$; moreover:

i) $\int 1 \cdot \ln(1+x)dx = x\ln(1+x) - \int x \cdot \dfrac{1}{1+x}dx = x\ln(1+x) - \int \dfrac{x+1-1}{1+x}dx =$

$= x\ln(1+x) - x + \ln(1+x)$

ii) $\int 1 \cdot \ln(1+3x)dx = x\ln(1+3x) - \int x \cdot \dfrac{3}{1+3x}dx = x\ln(1+3x) - \int \dfrac{3x+1-1}{1+3x}dx =$

$= x\ln(1+3x) - x + \frac{1}{3}\int \dfrac{3}{1+3x}dx = x\ln(1+3x) - x + \frac{1}{3}\ln(1+3x)$

if we call: $H(x) = x - (x\ln(1+3x) - x + \frac{1}{3}\ln(1+3x)) + x\ln(1+x) - x + \ln(1+x)$,

then applying Barrow's Rule we obtain: area (A)$= [H(x)]_0^{x^*} = H(x^*) - H(0)$ area units.