Universidad Carlos III de Madrid

Ejercicio	1	2	3	4	5	6	Total
Puntos							

Department of Economics Mathematics I Final Exam June 25th, 2015 Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = x \ln x$.**

(a) Draw the graph of the function, obtaining firstly its domain, the intervals where $f(x)$ inceases and decreases, its asymptotes and image.

(b) Consider the function $f(x)$ defined just in the interval $[1/e, \infty)$. Draw the graph of the function $f^{-1}(x)$, obtaining firstly its domain, image, the intervals where $f^{-1}(x)$ increases and decreases, the x-intercepts and y-intercepts and its fixed points.

Hint 1: Don't find the analytical expression of $f^{-1}(x)$.

Hint 2: The fixed points of $f(x)$ and $f^{-1}(x)$ are the same.

(a) **1 punto**

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a) The domain of f is the interval $(0, \infty)$. There are no vertical asymptotes, because at $x = 0$,

$\lim\limits_{x \longrightarrow 0+} f(x) = 0 \cdot (-\infty) = \lim\limits_{x \longrightarrow 0+} \dfrac{\ln x}{1/x} = \frac{\infty}{\infty} =$ L'Hospital's Rule $=$

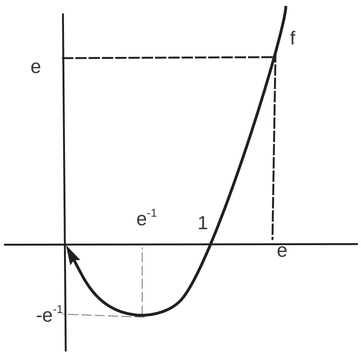
$= \lim\limits_{x \longrightarrow 0+} \dfrac{1/x}{-1/x^2} = \lim\limits_{x \longrightarrow 0+} (-x) = 0$

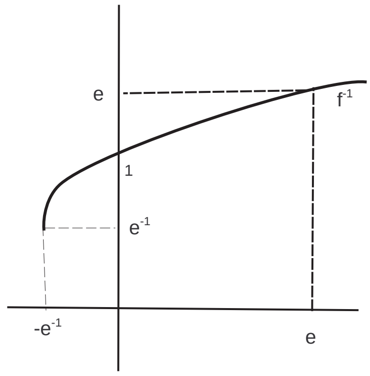
There are neither horizontal nor oblique asymptotes at ∞, because $\lim\limits_{x \longrightarrow \infty} \dfrac{f(x)}{x} = \lim\limits_{x \longrightarrow \infty} f(x) = \infty$.

On the other hand, since $f\,'(x) = 1 + \ln x$, and $1 + \ln x = 0 \Longleftrightarrow \ln x = -1 \Longleftrightarrow x = e^{-1} = 1/e$, we notice that $f\,'$ is increasing on $(0, \infty)$, is negative on $(0, 1/e)$ and positive on $(1/e, \infty)$ then we know that f is decreasing on the interval $(0, 1/e]$ and increasing on $[1/e, \infty)$.

Finally, since $f(1/e) = (1/e)\ln(1/e) = -1/e$, $\lim\limits_{x \longrightarrow \infty} f(x) = \infty$, $\lim\limits_{x \longrightarrow 0+} f(x) = 0$ and the function is continuous on $(0, \infty)$ then its range is $[-1/e, \infty)$.

So, we can conclude after all that the graph on the function $f(x)$ is the first figure:





b) Knowing that the function $f(x)$, is continuous and increasing on $[1/e, \infty)$ and its range $[-1/e, \infty)$ this means that the inverse function $f^{-1}(x)$ is increasing in its domain, the interval $[-1/e, \infty)$, and its range will be $[1/e, \infty)$.

Because the only x-intercept and y-intercept point of the restricted function is $(1, 0)$, then $f(1) = 0$, and the unique equivalent point of the inverse will be $(0, 1)$, so $f^{-1}(0) = 1$.

Finally, there is only one fixed point for both f and f^{-1}that is $x = e$, since $f(x) = x \Longleftrightarrow \ln x = 1 \Longleftrightarrow x = e$. So, the graph of the function $f^{-1}(x)$ will be approximately as you can see in the second figure.

(2) **Let** $y = f(x)$ **be an implicit function defined by the equation** $axy + 2 = 2xe^y + x^2y$**, in a neighborhood of the point** $(1, 0)$ **and** $a \neq 3$.

(a) Depending on the parameter a, calculate the derivative of $f(x)$ at $(1, 0)$.
What are the values of a, such that the function is increasing or decreasing near $x = 1$?

(b) Depending on the parameter a, find the tangent line of f at $(1, 0)$. Give the values of a, when that tangent line is parallel and perpendicular to the angle bisector of the first quadrant ($y = x$).

1 point

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a) Firstly, we compute the first derivative of the equation with respect to x,
$ay + axy' = 2e^y + 2xy'e^y + 2xy + x^2y'$
Next we substitute $x = 1, y = 0$ in order to obtain:
$ay' = 2 + 2y' + y' \iff (a-3)y' = 2 \iff y' = 2/(a-3)$.
Consequently, the implicit function will behave in a neighbourhood of $x = 1$ increasingly when $a > 3$ and decreasingly if $a < 3$.

b) The equation of the tangent line is:
$y - 0 = (2/(a-3))(x-1)$.
Then, that line will be parallel to $y = x$ if $\frac{2}{a-3} = 1 \iff a = 5$.
In the same way, that line will be parallel to $y = x$ if$\frac{2}{a-3} = -1 \iff a = 1$

(3) **Let** $C(x) = 2x^2 - 3x + C_0$ **be the cost function and** $p(x) = 197 - 2x$ **the (inverse) demand function of a monopolistic firm, that produces at least two units.**

(a) Find the production x_0 that maximizes the profit of the firm.

(b) Suppose that $x_1 = 2x_0$ is the production that minimizes the average cost for the company, find C_0.

Remark: Justify all your anwers.

1 point

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a) The profit function is:
$B(x) = I(x) - C(x) = 197x - 2x^2 - (2x^2 - 3x + C_0) = -4x^2 + 200x - C_0$
and its critical point x_0 is:
$B'(x_0) = -8x_0 + 200 = 0 \Longleftrightarrow x_0 = 25.$
Thus, because $B''(x) < 0$ in its domain, the profit function is strictly concave then the critical point is the unique global maximizer point for the profits.

b) Firstly, the average cost function is $\dfrac{C(x)}{x} = \dfrac{C_0}{x} - 3 + 2x$ and its first order derivative is
$(\dfrac{C(x)}{x})' = -\dfrac{C_0}{x^2} + 2$
Because $x_1 = 2x_0 = 50$ has to be a critical point of the average cost function:
$-\dfrac{C_0}{50^2} + 2 = 0 \Longleftrightarrow C_0 = 2.50^2 = 5.000.$
Then, because $(\dfrac{C(x)}{x})'' > 0$ the average cost function is strictly convex in its domain and the critical point will be the unique global minimizer for the function.

(4) **Let a, b be two real numbers, and consider the following piecewise function**

$$f(x) = \begin{cases} e^{a+x} & \text{if } x < -3 \\ 1 & \text{if } x = -3 \\ \sqrt{x+b} & \text{if } x > -3 \end{cases}.$$

(a) Discuss the continuity of the function f with respect to the numbers a, b.

(b) Discuss the differentiability of the function f with respect to the numbers a, b.

1 point

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a) For any value of a, the function is continuous on $x < -3$.
Furthermore, at $x = -3$, the function is continuous from the left if:
$\lim\limits_{x \longrightarrow -3^-} f(x) = f(-3) \Longleftrightarrow e^{a-3} = 1 \Longleftrightarrow a = 3.$
The function is continuous on $x > -3$ if $b \geqslant -3$.
And in particular, f is contiuous at $x = -3$ from the right if.
$\lim\limits_{x \longrightarrow -3^+} f(x) = f(-3) \Longleftrightarrow \sqrt{b-3} = 1 \Longleftrightarrow b = 4.$
Consequently, $f(x)$ is continuous for every x when $a = 3, b = 4$.
Finally, we can notice that for those values $a = 3, b = 4$, the function is continuous for every point in its domain.

b) Obviously, when $x \neq -3$ our function is differentiable in its domain, because the function is an exponential or a square root function and both are differentiable.
At the point $x = -3$, we need to calculate the one-sided derivatives, knowing that the function is continous at the point when $a = 3, b = 4$.
$f'_-(-3) = \lim\limits_{x \longrightarrow -3^-} f\,'(x) = \lim\limits_{x \longrightarrow -3^-} e^{x+3} = e^0 = 1$
$f'_+(-3) = \lim\limits_{x \longrightarrow -3^+} f\,'(x) = \lim\limits_{x \longrightarrow -3^+} \dfrac{1}{2\sqrt{x+4}} = \frac{1}{2}.$
Then the function will never be differentiable at $x = -3$.

(5) **Consider the set of points in the plane A bounded by the curves $y = \ln(x+4), y = -e^x$ and the straight lines $x = 0,\ x = 2$.**

(a) Draw the set A and find, if they exist, the maximals, minimals, maximum and minimum points of A.

(b) Calculate the area of the region A.

Hint 1: Pareto order is defined by: $(x_0, y_0) \leq_P (x_1, y_1) \iff x_0 \leq x_1$ and $y_0 \leq y_1$.

Hint 2: Do not calculate the values either of the logarithms (ln) or of the exponentials (exp) in the solutions.

1 point

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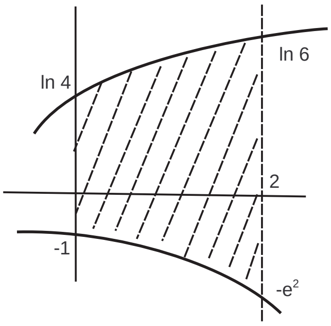
a) The function $f(x) = \ln(x+4)$ is positive and increasing on the interval $[0, 2]$,
and the function $g(x) = -e^x$ is negative and decreasing on the real line,
so the set A has a shape which is approximately like this:



Clearly, we can deduce looking at the drawing that
$\{\text{maximals of } (A)\} = \{\text{maximum of } (A)\} = \{(2, \ln 6)\}$
$\{\text{minimals of } (A)\} = \{(x, y) : 0 \leq x \leq 2, y = -e^x\}$; the minimum of (A) doesn‘t exist.

b) The required area is below the logarithim function and above the exponential one and inbetween the vertical lines $x = 0, x = 2$.

So, the area is: $\int_0^2 (\ln(x+4) + e^x)dx$

Integrating by parts:

$\int \ln(x+4)dx = x\ln(x+4) - \int \dfrac{x}{x+4}dx = x\ln(x+4) - \int \frac{x+4-4}{x+4}dx =$
$= x\ln(x+4) - x + 4\ln(x+4) = (x+4)\ln(x+4) - x.$

And applying Barrow‘s Rule we obtain:

$[(x+4)\ln(x+4) - x + e^x]_0^2 = 6\ln 6 - 2 + e^2 - (4\ln 4 + 1) =$
$= 6\ln 6 + e^2 - 4\ln 4 - 3$ area units.

(6) **Given the function** $f(x) = \dfrac{x}{\sqrt{x-1}}$ **defined in the interval** $(1, \infty)$.

(a) Find the primitive function $F(x)$ of f such that satisfies $F(2) = \frac{8}{3}$.

(b) Prove the inequality $\sqrt{x+1} < \dfrac{x}{\sqrt{x-1}}$ and use it to find a lower bound of the definite integral

$$\int_3^8 \frac{x}{\sqrt{x-1}} dx.$$

1 point

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a) Making the change of variable $x - 1 = t^2, dx = 2tdt$, we can deduce that the primitive function of f will be:
$F(x) = \int \dfrac{x}{\sqrt{x-1}} dx = \int \dfrac{1+t^2}{t} 2tdt = 2\int(1+t^2)dt = 2(t + t^3/3) + C =$
$= 2\sqrt{x-1} + \frac{2}{3}\sqrt{(x-1)^3} + C$
Because $\frac{8}{3} = F(2) = \frac{8}{3} + C$, we know that $C = 0$.
So, the answer is $F(x) = 2\sqrt{x-1} + \frac{2}{3}\sqrt{(x-1)^3}$.

b) Multipling the inequality by the denominator we get the equivalent inequation:
$\sqrt{x+1}\sqrt{x-1} < x$, squaring both sides we obtain:
$(x+1)(x-1) = x^2 - 1 < x^2$
so, we can see that the inequality is satisfied. From that fact we can deduce that
$\int_3^8 \sqrt{x+1} dx < \int_3^8 \dfrac{x}{\sqrt{x-1}} dx$
Thus, because the value of the first integral is:
$\int_3^8 \sqrt{x+1} dx = [\frac{2}{3}(x+1)^{3/2}]_3^8 = \frac{2}{3}(27-8) = \frac{38}{3}$,
we can state that $\frac{38}{3} < \int_3^8 \dfrac{x}{\sqrt{x-1}} dx$.