Universidad Carlos III de Madrid

Economics Department **Final Exam of Mathematics I** **June 17th, 2013**

Time Length: 2 hours.

FAMILY NAMES:		**FIRST NAME:**
DNI:	**DEGREE:**	**GROUP:**

(1) **Let the function** $f(x) = \dfrac{1}{e^x + e^{-x}}$**. We ask the following:**

(a) Represent the graph of $f(x)$ obtaining previously the domain, the asymptotes, the intervals of increasing and decreasing, and the image of $f(x)$.

(b) Consider the function $f(x)$ restricted to the interval $[0, \infty)$.
Draw the graph of $f^{-1}(x)$, obtaining previously the domain, the image, the intervals of increasing and decreasing and the asymptotes of $f^{-1}(x)$.
Hint: don't try to obtain the analytical form of $f^{-1}(x)$.
1 point

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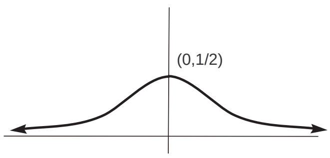
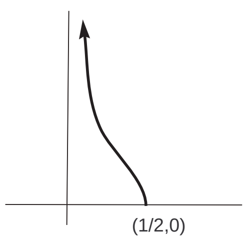
a) The domain is the real line, because the denominator of the function is always positive.
There are no vertical asymptotes, as the function is always continuous.
Nevertheless, there are horizontal asymptotes, because
$\lim_{x \longrightarrow \pm\infty} f(x) = \frac{1}{\infty} = 0$.
On the other hand, as $f\,'(x) = \dfrac{-e^x + e^{-x}}{(e^x + e^{-x})^2}$, and the exponential function is increasing, you can deduce that f is increasing on the interval $(-\infty, 0]$ and decreasing on $[0, \infty)$.
Finally, as $f(0) = \frac{1}{2}$, $\lim_{x \longrightarrow \pm\infty} f(x) = 0$ and the function is continuous and monotone on the intervals $(-\infty, 0]$ and $[0, \infty)$, the image is $(0, \frac{1}{2}]$.
So, the graph of the function $f(x)$ will be, approximately, this way:



b) We consider the function $f(x)$, continuous and decreasing on $[0, \infty)$ and with image $(0, \frac{1}{2}]$.
For that reason, the inverse function will have as domain the interval $(0, \frac{1}{2}]$ and its image will be the interval $[0, \infty)$.
The inverse function will be decreasing and it will have a vertical asymptote towards ∞ at 0^+, by symmetry with respect to the main diagonal.
So, the graph of the function $f^{-1}(x)$ will be, approximately, in this way:



(2) **Let** $f(x) = 8xe^{-x^2}$**. We ask the following:**

(a) Compute the intervals where the function is increasing or decreasing, the local and/or global extrema, and the asymptotes.

(b) Compute the intervals where the function is concave or convex, inflection points, and draw the graph of $f(x)$.

1 point

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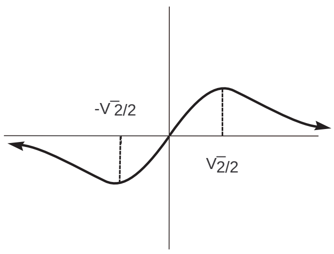
a) First of all, the derivative of the function is:
$f\,'(x) = 8e^{-x^2} - 16x^2e^{-x^2} = 8e^{-x^2}(1-2x^2)$.
So, the function is increasing on the interval $[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and
decreasing on the intervals $(-\infty, -\frac{\sqrt{2}}{2}]$ and $[\frac{\sqrt{2}}{2}, \infty)$.
As the function is continuous on the real line, it has no vertical asymptote.
On the other hand, as $\lim_{x\longrightarrow\pm\infty} f(x) = \lim_{x\longrightarrow\pm\infty} \frac{8x}{e^{x^2}} = (L'Hopital) = \lim_{x\longrightarrow\pm\infty} \frac{8}{2xe^{x^2}} = 0$
the function has a horizontal asymptote $y = 0$ at $\pm\infty$.
Finally, as $f(0) = 0$, the function reaches a local and global minimum at $x = -\frac{\sqrt{2}}{2}$
and a local and global maximum at $x = \frac{\sqrt{2}}{2}$.

b) Now, we compute the second derivative of the function. As
$f"(x) = -16xe^{-x^2}(3-2x^2)$
we can deduce that f is convex on the intervals $[-\frac{\sqrt{3}}{\sqrt{2}}, 0]$ and $[\frac{\sqrt{3}}{\sqrt{2}}, \infty)$, where $f"$is positive.
On the other hand, f is concave on the intervals $(-\infty, -\frac{\sqrt{3}}{\sqrt{2}}]$ and $[0, \frac{\sqrt{3}}{\sqrt{2}}]$, where $f"$is negative.
So, the function has inflection points at $x = -\frac{\sqrt{3}}{\sqrt{2}}, 0, \frac{\sqrt{3}}{\sqrt{2}}$.
For these reasons, the function has a graph, approximately, in this way:



(3) **Let $B'(x) = 30 - 2x$ the function of marginal profits and $C_0 = 100$ the fixed costs of a monopolist company. We ask:**

(a) Find the production x_0 that maximizes the profit of this company. Find the profits obtained and the profit by unity for that quantity x_0.

(b) Find the production x_1 that maximizes the profit by unit of that company. Find the profits and the profits by unit for that quantity x_1.

Remark: justify the answers.

1 point

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a) As the profits function is $B(x) = -x^2 + 30x - 50$, the production x_0 that maximizes the profits will be:

$B'(x_0) = -2x_0 + 30 = 0 \Longleftrightarrow x_0 = 15$.

And, observing that the profits function is concave, as $B"(x) < 0$,

the critical point will be the unique global maximizer of the profits function.

For that production, the profits will be: $B(15) = -15^2 + (30).15 - 100 = -225 + 450 - 100 = 125$

And the profits by unity will be: $\dfrac{B(15)}{15} = \dfrac{125}{15} = \dfrac{25}{3} \approx 8,3$

b) First of all, as $\dfrac{B(x)}{x} = -x + 30 - \dfrac{100}{x}$. If we derivate this function, we obtain:

$(\dfrac{B(x)}{x})' = -1 + \dfrac{100}{x^2} = 0 \Longleftrightarrow x^2 = 100$, so the only critical point is $x_1 = 10$.

And, observing that the function of mean profits is concave, as $(\dfrac{B(x)}{x})" < 0$

we obtain that the critical point will be the unique global maximizer of the mean profits.

For this level of production, the profis will be: $B(10) = -10^2 + (30).10 - 100 = -100 + 300 - 100 = 100$

And the mean profits will be: $\dfrac{B(10)}{10} = \dfrac{100}{10} = 10$

ANNEX for problems 1, 2 and 3

(4) **Let $y = f(x)$ the function defined implicitly, near the point $x = 1, y = 0$ by the equation $axy - 2e^y = 2x - 4$. We ask the following:**

(a) Find a such that the slope of the tangent line to the function $y = f(x)$ at the point $(1, 0)$ will be equal to -2.

(b) Find $f"(1)$ and draw approximately the graph of the function near $x = 1, y = 0$.

Hint for b): it is only necessary to compute the sign of $f"(1)$.

1 point

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a) Let's compute the first derivative of this function:

$ay + axy' - 2y'e^y = 2$;

Now, substituting $x = 1, y = 0$ in the previous equation:

$(a-2)y' = 2 \Longrightarrow y' = \dfrac{2}{a-2} = -1 \Longrightarrow a = 1$

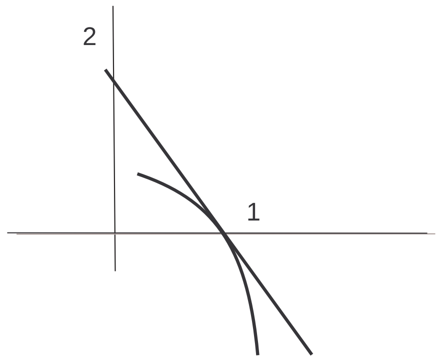
b) If we derive again the implicit equation, we obtain:

$2y' + xy" - 2[(y')^2 + y"]e^y = 0$

Substituting $x = 1, y = 0, y' = -1$ in the previous equation, we obtain:

$-12 - y" = 0 \Longrightarrow y" = -12$ so the function is concave near $x = 1$.

So, the graph of the function near the point $(1, 0)$ will be, approximately, this way:



(5) **Let's consider the set** $A = \{(x, y) : x^2 \le y \le 2 - |x|\}$. **We ask:**

(a) Represent the set A and find the maximal and minimal points, maximum and minimum, of A, if they exist.

(b) Compute the area of this region.

Hint: Pareto order is defined by: $(x_0, y_0) \le_P (x_1, y_1) \Longleftrightarrow x_0 \le x_1, y_0 \le y_1$.

1 point

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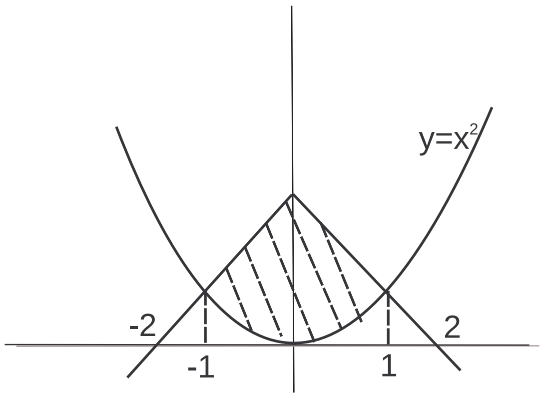
a) As $2 - |x| = 2 + x$ if $x \le 0$ and $2 - |x| = 2 - x$ if $x \geqslant 0$, we obtain the intersection points of $2 - |x|$ and x^2.

$x^2 = 2 + x, x \le 0 \Longleftrightarrow x = -1 \Longrightarrow y = 1$

$x^2 = 2 - x, x \geqslant 0 \Longleftrightarrow x = 1 \Longrightarrow y = 1$

So, the region bounded by the graphs of these functions has a shape, approximately, in this way:



Obviously, $\{\text{maximal points}(A)\} = \{(x, y) : y = 2 - x, 0 \le x \le 1\}$ $\Longrightarrow$it doesn't exist the maximum.

$\{\text{minimal points}(A)\} = \{(x, y) : y = x^2, -1 \le x \le 0\}$ $\Longrightarrow$it doesn't exist the minimum.

b) El área solicitada, observando la simetría respecto al eje vertical, es:

$2\int_0^1 (2 - x - x^2)dx = 2[2x - x^2 - \frac{x^3}{3}]_0^1 = \frac{7}{3}$ area units.

(6) **Given the function** $f(x) = \dfrac{8-6x}{x^2-4}$, **we ask:**

(a) Find the equation of the primitive $F(x)$ of $f(x)$ that satisfies $F(3) = 0$.

(b) Find the tangent line to F at the point $a = 3$, and use it to obtain an approximation of the value $F(2'9)$.

1 point

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a) As $\dfrac{8-6x}{x^2-4} = \dfrac{A}{x-2} + \dfrac{B}{x+2} = \dfrac{A(x+2)+B(x-2)}{(x-2)(x+2)} \Longrightarrow 8-6x = A(x+2)+B(x-2)$, we deduce that

taking $x = -2 \Longrightarrow B = -5$; taking $x = 2 \Longrightarrow A = -1$.

So $F(x) = \int f(x)dx = -\ln|x-2| - 5\ln|x+2| + C$.

And, as $F(3) = -5\ln 5 + C = 0 \Longrightarrow C = 5\ln 5$

So the equation of the primitive is $F(x) = 5\ln 5 - \ln|x-2| - 5\ln|x+2|$.

b) The equation of the tangent line to F at the point $a = 3$ is:

$y = F(3) + F'(3)(x-3)$

As $F(3) = 0, F'(3) = f(3) = -\frac{10}{5} = -2$, we deduce that

$y = -2(x-3)$.

So $F(2'9) \approx -2((2'9-3) = 0'2$

ANNEX for problems 4, 5 and 6