| Question | 1 | 2 | 3 | 4 | 5 | 6 | Grade |
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Universidad Carlos III de Madrid

**Departamento de Economía** **Final exam, Mathematics I** **June 16, 2009**

| **LAST NAMES:** | | **NAME:** |
|---|---|---|
| **DNI:** | **Title:** | **Group:** |

1. **Given function $f(x) = e^{x+1} - 2$, we ask:**
   a) Find the inverse function of $f(x)$ and represent it (the inverse function). In other words, find the domain, the range (image), the axes cuting points, and the increasing/decreasing interval of $f^{-1}$.
   b) Represent the graph of function $g(x) = |f(x)|$. In other words, find the domain, the range (image), the axes cuting points and the increasing(decreasing intervals of $g$.
   **1 point**

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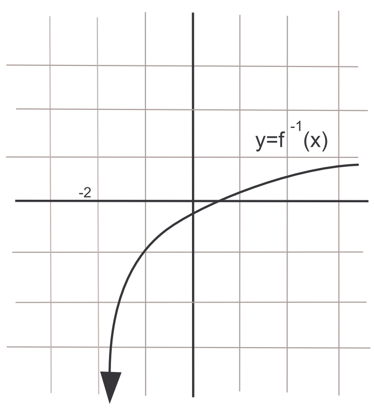
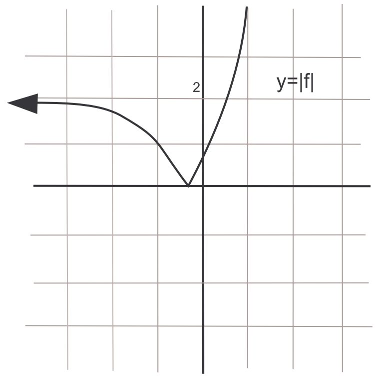
a) In order to find the inverse function, we set the equation $y = f^{-1}(x)$, which will satisfy: $e^{y+1} - 2 = f(y) = ff^{-1}(x) = x$, so taht $x+2 = e^{y+1}$, and then $\ln(x+2) = y+1$; hence $f^{-1}(x) = y = \ln(x+2)-1$. To represent $f^{-1}$, it is enough to notice that $f$ is increasing, has a horizontal asymptote $y = -2$ at $-\infty$, that $\lim_{x\to\infty} f(x) = \infty$, and that it cuts the axes at the following points: horizontal axis at point (ln2-1,0); vertical axis at point (0,e-2).
By symmetry, the domain of $f^{-1}$ is the interval $(-2, \infty)$, the image is $\mathbb{R}$, the axes cuting points are: horizontal axis at point $(e-2, 0)$ ; vertical axis at point $(0, ln2-1)$, $f^{-1}$ is increasing in all its domain. The sketch of the graph of $f^{-1}(x)$ is:




Remark: We can also study the graph of $f^{-1}$ directly observing $f^{-1}(x) = \ln(x+2) - 1$, instead of using symmetry.

b) As it is obvious, the domain of $g$ is $\mathbb{R}$. Regarding the image of $g$, it is clear that it is included in the interval $[0, \infty)$.
On the other hand, since $g(\ln 2 - 1) = 0$, $\lim_{x\to\infty} g(x) = \infty$, and $g$ is continuous, we infer that $Im(g) = [0, \infty)$.
The axis cutting points are the same as the function $f$. In other words, $(ln2-1, 0)$ and $(0, e-2)$.
Finaly, $f$ is decreasing in the interval $(-\infty, \ln 2 - 1)$ and increasing in the interval $(\ln 2 - 1, \infty)$.
Hence, the graph of $g$ is, more or less:


2
y=|f|

**2. Let $a, b$ be real numbers, and consider the function defined by**

$$f(x) = \begin{cases} |x+a|, & x < 2 \\ b, & x = 2 \\ -x^2 + 6x, & x > 2 \end{cases}$$

a) Study, according to the values of $a$ and $b$, the continuity of $f$ in the inteval $[1, 3]$.

b) State the theorem by Weierstrass and, assyming that $a = -10$, determine, depending on the values of $b$, when are true the hypotheses and the thesis (assumptions and conclusion, respectively) of that theorem for function $f$ in the interval $[0, 2]$.

**1 point**

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a) In order for $f$ to be continuous at $2^+$ it is necessary and sufficiente that $b = 8$.
Now, in order for $f$ to be continuous at $2^-$, it is necessary and sufficient that $|2 + a| = 8$; or, equivalently, that $a = 6$ or $a = -10$.

b) If $b = 8$, the conditions and conclusions of the theorem, by the previous item.
If $b > 8$, the conditions are not fulfilled (by the previous item), and the conclusion is not fulfilled either, since
$\lim\limits_{x \longrightarrow 2^-} f(x) = 8, f(2) = b > 8$, hence $\inf\{f(x) : x \in [0, 2]\} = 8 \neq f(x)$
because $f(x)$ is decreasisng in $[0, 2)$.
Now, if $b < 8$, the conditions are not met (by the previous item), but the conclusion is indeed satisfied, since
$\max\{f(x) : x \in [0, 2]\} = f(0)$, $\min\{f(x) : x \in [0, 2]\} = f(2)$.

**3. Let $f(x) = xe^{-x}$. Regarding $f$, we ask:**

a) Find when it is increasing/decreasing, find the local and/or global extrema, and its asymptotes

b) Find when it is concave or convex, find its inflexion points, and sketch a graph of it.

**1 point**

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a) Deriving the function, we get that $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$. Then, $f'(x) = 0 \Longleftrightarrow e^{-x} = xe^{-x} \Longleftrightarrow x = 1$. Since $f'(0) = 1, f'(2) < 0$, we get:

$f$ is increasing in $(-\infty, 1)$ and decreasing in $(1, \infty)$; $x = 1$ is a local and global maximum of $f$.

On the other hand, since $\lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} e^{-x} = \infty$, $f$ does not have either horizontal or oblique asymptotes at $-\infty$.

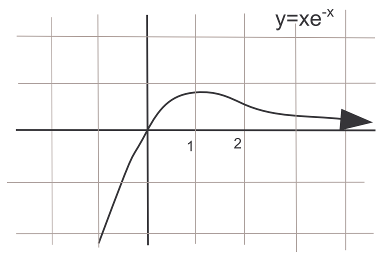
Nevertheless, since $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{e^x} =$ ( by L'Hopital's rule) $= \lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$, $f$ has horizontal asymptote $y = 0$ at $\infty$.

Finaly, since the funtion is continuous, it does not have a vertical asymptote.

b) $f"(x) = -e^{-x} + (1-x)e^{-x}(-1) = (x-2)e^{-x}$. Hence:

$f$ is concave in interval $(-\infty, 2)$ and convex in $(2, \infty)$. The only inflexion point if found in $x = 2$.

Adding that $f(0) = 0$, $f$ is negative at negative points, and potitive at positive points, the sketch of $f$ is



**4. Let $C(x) = 30 + 100x + 0.'3x^2$ be the cost function of a monopoly, where $x \geq 0$ is the number of units produced of a given good. The inverse demand function (or unit price) is $p(x) = 500 - 0'7x$. We ask:**

a) Prove that the benefits function is concave. From that, determine the quantity $x$ that maximizes benefits.

b) Prove taht the mean cost function is convex. From that, determine the quantity $x$ that minimizes mean cost.

**1 point**

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a) The benefits function is: $B(x) = p(x)x - C(x) = 500x - 0'7x^2 - (30 + 100x + 0'3x^2) =$
$= -x^2 + 400x - 30$ , so that $B"(x) = -2 < 0$.
Hence $B(x)$ is concave, and the critical point is the unique, global, maximizing pointl Now:
$B\ '(x) = -2x + 400 = 0 \Longrightarrow x = \dfrac{400}{2} = 200$
which is acceptable, since it is positive. Its selling price is
$p(200) = 500 - (0'7).200 = 360 > 0$.

b) The mean cost function is convex, since $C_{med}(x) = \dfrac{C(x)}{x} = \dfrac{30}{x} + 100 + 0'3x$ and
$C"_{med}(x) = \dfrac{60}{x^3} > 0$. Hence, the cricial point is the unique global minimizing point. Now
$C'_{med}(x) = -\dfrac{30}{x^2} + 0'3 = 0 \Longrightarrow x = 10$. So, at that point, we have the minimum of the mean cost function.

**5. Given $f(x) = x\ln(0'25x)$, defined in the interval $(0,\infty)$, we ask:**

a) Analyze when a given primitive $F$ of $f$ is increasing/decreasing, and when it is concave/convex. Also, analyze the existence of local and/or global extrema for $F$.

b) Find the primitive $F$ of $f$ which satisfies $F(4) = 0$.

Remark: it is not necessary to know the form of $F(x)$ in order to answer item a)

**1 point**

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a) First of all, we need $F'$. But $F'(x) = f(x) = x\ln(0'25x)$

Then $F$ is increasing when $F'$ is positive, which occurs when $0'25x > 1$, in other words, in the interval $(4,\infty)$.

By the same token, $F$ is decreasing when $F'$ is negative, which occurs in the interval $(0,4)$.

Hence, $F$ reaches a global and local minimum at $x = 4$.

b) First of all, using the method of integration by parts, we find the equation of a representative primitive of $f$:

$\int f(x)dx = \int x\ln(0'25x)dx = \dfrac{x^2}{2}\ln(0'25x) - \int \dfrac{x^2}{2}\dfrac{0'25}{0'25x}dx = \int \dfrac{x}{2}dx = \dfrac{x^2}{2}\ln(0'25x) - \dfrac{x^2}{4} + C$

Now, we find the primitive that satisfies $F(4) = 0$, and for that, we set $8\ln(1) - 4 + C = F(4) = 0 \Longrightarrow C = 4$.

Hence, $F(x) = \dfrac{x^2}{2}\ln(0'25x) - \dfrac{x^2}{4} + 4$.

**6. Consider the set $A = \{(x,y) : x - 3 \leq y \leq -x^2 + 3x\}$. We ask:**

a) Find the area of the set $A$.

b) Using Pareto ordering, determine the set of maximal and minimal points of $A$, as well as its maximum and minimum points, if they exist.

Hint: do the graph of $A$.

**1 point**

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a) Since the parable $y = -x^2 + 3x$ cuts the line $y = x - 3$ at points $x = -1, x = 3$, and it is concave, it is above that line in the interval $[-1, 3]$.

Hence, the area of $A$ is:

$\int_{-1}^{3}((-x^2 + 3x) - (x-3))dx = \int_{-1}^{3}(-x^2 + 2x + 3)dx = [(-\frac{1}{3})x^3 + x^2 + 3x]_{-1}^{3} =$

$= (-9 + 9 + 9) - (\frac{1}{3} + 1 - 3) = 9 - (-\frac{5}{3}) = \frac{32}{3}$.

b) The parable reaches a maximum at $x = \frac{3}{2}$. This follows, since with $f(x) = -x^2 + 3x$ we have $f\,'(x) = -2x + 3 = 0 \Longleftrightarrow x = \frac{3}{2}$, and the parable is decreasing in the interval $[\frac{3}{2}, 3]$.

Since the points $(x, y)$ of the set $A$ satisfy $x \leq 3$, we infer that $Maximals(A) = \{(x,y) : y = -x^2 + 3x, \frac{3}{2} \leq x \leq 3\} \Longrightarrow$. There is no maximum point for $A$.

On the other hand, since the parable $y = -x^2 + 3x$ and the line $y = x - 3$ are increasing in the interval $[-1, \frac{3}{2}]$, and the points $(x, y)$ of $A$ satisfy $x \geq -1$, we infer that $Minimals(A) =$; $Minimum(A) = (-1, -4)$.

Remark: the set $A$ has the following graph: