Question	1	2	3	Part I
Points				

Part I	Part II	Class grade	Final grade

Universidad Carlos III de Madrid

Departamento de Economía **Final Exam Mathematics I** **January 14, 2009**

Last names:		**First Name:**
DNI:	**Title:**	**Group:**

1. Set $f(x)=\dfrac{e^x}{x-1}$**. We ask:**

a) Find the intervals in which funtion f is increasing/decreasing, and the local and/or global extrema.

b) Find the asymptotes of f.

1 punto

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a) To see where f is increasing, we find f' and study its sign: $f'(x)=(\frac{e^x}{x-1})'=\frac{e^x(x-1)-e^x}{(x-1)^2}=\frac{e^x(x-2)}{(x-1)^2}$. Its sign comes from the sign of $x-2$ since e^x and $(x-1)^2$ are always positive. So we get:
$f'>0 \Leftrightarrow x\in(2,\infty)$, and hence f is increasing on $(2,\infty)$.
$f'<0 \Leftrightarrow x\in(-\infty,1)\cup(1,2)$, and hence f is decreasing on $(-\infty,1)\cup(1,2)$.
Regarding local extrema, we know that f reaches a local minimum at $x=2$, since here the necesary condition $f'(2)=0$ is satisfied, as well as the sufficient condition of change of sign of f'.
Regarding global extrema, $x=2$ can not be a global minimum, since $f(2)>0>f(0)$; given that $\lim_{x\to 1^+}\frac{e^x}{x-1}=\infty$, it is clear that f does not have a global maximum.

b) To find the vertical asymptotes, notice that the domain of f is $\mathbb{R}-\{1\}$. The right-side limit is $\lim_{x\to 1^+}\frac{e^x}{x-1}=\frac{e}{0^+}=\infty$. The left-side limit is $\lim_{x\to 1^-}\frac{e^x}{x-1}=\frac{e}{0^-}=-\infty$. So $x=1$ is a vertical asymptote of f.
To study horizontal asymptotes, taking the limit of f when x tends to ∞, we get that $\lim_{x\to\infty}\frac{e^x}{x-1}=\frac{e^\infty}{\infty}=\frac{\infty}{\infty}=?$. Using L'Hopital, deriving numerator and denominator, $\lim_{x\to\infty}\frac{e^x}{x-1}=\lim_{x\to\infty}\frac{e^x}{1}=\frac{\infty}{1}=\infty$. Hence, f does not have a horizontal asymptote at infinity.
Now, taking the limit at $-\infty$ we get $\lim_{x\to-\infty}\frac{e^x}{x-1}=\frac{e^{-\infty}}{-\infty}=\frac{0}{\infty}=0$. So $y=0$ is a horizontal asymptote at minus infinity.
Let us check now oblique asymptote at ∞. For that, we find $\lim_{x\to\infty}\frac{f(x)}{x}=\lim_{x\to\infty}\frac{\frac{e^x}{x-1}}{x}=\lim_{x\to\infty}\frac{e^x}{x^2-x}=\frac{\infty}{\infty}=?$. Applying L'Hopital two times, we get $\lim_{x\to\infty}\frac{e^x}{x^2-x}=\lim_{x\to\infty}\frac{e^x}{2x-1}=\lim_{x\to\infty}\frac{e^x}{2}=\frac{\infty}{2}=\infty$. Hence, f does not have oblique asymptotes.

$$f(x) = \begin{cases} ax^2+3, & x<1 \\ b, & x=1 \\ \dfrac{c}{x}, & x>1 \end{cases}$$

a) Analyze, according to the values of a, b and c, the derivability of f in the interval $(0,2)$.

b) State the mean value theorem, and find the values of a, b and c such that we can apply that theorem to function f in the interval $[0,2]$.

1 punto

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a) First of all, let's see if the function is continuous in the interval $(0,2)$. We observe that the only troublesome point is $x=1$. $f(x)$ is continuous at that point when:
$\lim_{x\longrightarrow 1^-} f(x) = f(1) = \lim_{x\longrightarrow 1^+} f(x)$, in other words, when $a+3=b=c$.
Now, asumming that $f(x)$ is continuous at $x=1$, $f(x)$ is derivable there when
$\lim_{x\longrightarrow 1^-} f'(x) = \lim_{x\longrightarrow 1^+} f'(x)$, in other words, when $2a=-c$.
Hence, $a+3=c=-2a$, or $a=-1, b=c=2$.
It follows that $f(x)$ is derivable in $(0,2)$ when $a=-1, b=c=2$.

b) The mean value theorem for funciton f defined in $[A,B]$ says:
If f is continuous in $[A,B]$ and derivable in (A,B), there is a poin $C\in(A,B)$ so that
$f(B)-f(A)=f'(C).(B-A)$.
In our case, for the theorem to be valid for f in the interval $[0,2]$,
it is enough that f is derivable in $(0,2)$, since in that case it follows directly that f is continuous there. In any case, f is continuous at $x=0$ and at $x=2$.
We then can apply the theorem when $a=-1, b=c=2$.

$(0, 1)$. **We ask:**

a) Find, through $f\,'(0)$, the equation of the tangent line to the graph of f at point $x = 0$, $y = 1$.

b) Approximate, using the tangent line equation, the value of $f(-0.1)$.

1 punto

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a) Deriving both sides of the equation, we get $\dfrac{1+y'}{x+y} + 2y' = 4$. Setting $x = 0, y = 1$ in that resulting equation, we get $\dfrac{1+y'}{0+1} + 2y' = 4$, so $f\,'(0) = y' = 1$.
Hence, the equation of the tangent line will be $y - 1 = 1.(x - 0)$, in other words, $y = x + 1$.

b) $f(-0.1) \approx y(-0.1) = -0.1 + 1 = 0.9$

Question	4	5	6	Part II
Points				

Part I	Part II	Class grade	Final grade

Universidad Carlos III de Madrid

Departamento de Economía **Final exam, Mathematics I** **January 14, 2009**

LAST NAMES:		**NAME:**
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1.

4. **Let $C(x) = C_0 + 60x + 0.01x^2$ be the cost function of a monopoly, where $C_0 \geq 0$, and where $x \geq 0$ is the quantity of units produced by the firm. The inverse demand function (or unit price function) is $p(x) = 90 - 0.02x$. We ask:**

   a) Prove that the benefit function is concave. From that, determine que quantity that maximizes benefits.

   b) For what values of C_0 it is true that the production that maximizes benefits coincide with the production that minimizes medium costs (unit costs)?

   1 point

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a) The benefits function is: $B(x) = p(x).x - C(x) = 90x - 0.02x^2 - (C_0 + 60x + 0.01x^2) =$
$= -0.03x^2 + 30x - C_0$, hence $B"(x) = -0.06 < 0$.
So $B(x)$ is concave. Because of that, the critical point is unique, and global maximum. Now:
$B\,'(x) = -0.06x + 30 = 0 \Longrightarrow x = \dfrac{30}{0.06} = 500$
which is acceptable, since it is a positive quantity, whose selling price should be
$p(500) = 90 - (0.02).500 = 80 > 0$.

b) The mean cost function is convex, since $C_{med}(x) = \dfrac{C(x)}{x} = \dfrac{C_0}{x} + 60 + 0.01x$ which satisfies
$C"_{med}(x) = \dfrac{2C_0}{x^3} > 0$. Hence, the critical point is unique, and is a global maximum. Now:
$C'_{med}(x) = -\dfrac{C_0}{x^2} + 0.01 = 0 \Longrightarrow x = \sqrt{100C_0}$. Hence, this point will coincide with the global maximizer when
$500 = \sqrt{100C_0} \Longrightarrow C_0 = \dfrac{500^2}{100} = 2500$.

$1 + x$,

a) Analyze when a given primitive F of f is increasing/decreasing, and when it is concave/convex. Also, analyze the existence of local and/or global extrema and inflexion points for F.

b) Find the primitive F of f which satisfies $F(0) = 1$.

Remark: it is not necessary to know the form of $F(x)$ in order to answer item a).

1 point

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a) First we need the derivative of F. Since

$F'(x) = f(x) = \dfrac{1-x}{1+x}$,

F is increasing when F' is positive, that is, in the interval $[0, 1)$.

By the same token, F is decreasing when F' is negative, that happens in the interval $(1, \infty)$.

Hence, F reaches a local and global maximum at the point $x = 1$.

In order to study concavity/convexity, as well as inflexion points of F, we need to know the second derivative of F. Now,

$F''(x) = f\,'(x) = \dfrac{-2}{(1+x)^2} < 0$

Then, F is concave in its entire domain, which is the interval $[0, \infty)$.

It follows that F does not have any inflexion point.

b) First of all, let us find the equation of a general primitive of f.

$\int f(x)dx = \int \dfrac{1-x}{1+x} dx = \int \dfrac{-1-x+2}{1+x} dx = \int (-1 + \dfrac{2}{1+x}) dx = -x + 2\ln(1+x) + C$

Now, we are interested in the primitive that satisfies $F(0) = 1$. So

$-0 + 2\ln(1+0) + C = F(0) = 1 \Longrightarrow C = 1$.

Hence our primitive is, $F(x) = -x + 2\ln(1+x) + 1$.

$x = 2$, **and the line** r, **which is tangent to the graph of** f **at** $c = 1$. **We ask:**

a) Represent the graph of the set A.

b) Find the Area of A.

Suggestion: show that r cuts the graph of f only at the tangency point.

1 point

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a) Since $f\,'(x) = 3x^2 - 6x \Longrightarrow f\,'(1) = 3 - 6 = -3$.

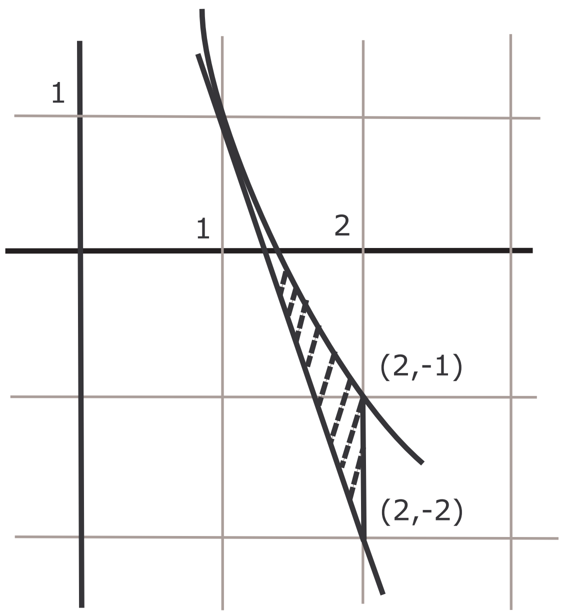
Hence, the equation of the tangent line to the graph of f at $c = 1$ is:

$y - 1 = (-3)(x - 1)$, es decir, $y = -3x + 4$.

On the other hand, in the interval $(1, 2]$ the graph of f is always above the tangent line we just found. This can be seen, for example, using the fact that f is convex in that interval, since $1 < x \leq 2 \Longrightarrow f"(x) = 6x - 6 > 0$.

On top of that, since $f\,'(x) = 3x^2 - 6x = 3x(x - 2) < 0$ if $1 < x < 2$, we infer that f is decreasing in that interval.

Finaly, observe that the tangent line cuts the tangent line and the graph of f at the points $(2, -2)$ and $(2, -1)$ of the vertical line $x = 2$, respectively. We can represent the graph of A aproximately as:



b) Since the graph of f is always above the tangent line r in the interval $(1, 2]$,

$Area(A) = \int_1^2 (f(x) - r(x))dx = \int_1^2 (x^3 - 3x^2 + 3 - (-3x + 4))dx =$

$= \int_1^2 (x^3 - 3x^2 + 3x - 1)dx = \int_1^2 (x - 1)^3 dx = [\,\dfrac{(x-1)^4}{4}]_1^2 = \frac{1}{4}$.