Universidad Carlos III de Madrid

Exercise	1	2	3	4	5	Total
Points						

Department of Economics **Mathematics I Final Exam** **January 11th 2024**

Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**	
ID:	**DEGREE:**	**GROUP:**	

(1) **Let $p(x) = 13 - \beta x$ be the inverse demand function and $C(x) = 16 + x + x^2$ be the cost function of a monopolistic firm, with $\beta > 0$.**

(a) Calculate the value of β such that the firm's profits are maximized at $x^* = 3$.

(b) Show that the minimum average cost is obtained at a higher level of production $\widetilde{x}$.

(c) If the regulator forces the firm to produce at its minimum average cost, with the value of β found in part (a), what will be the compensation that the firm will demand?

0.4 points part a); 0.4 points part b); 0.2 points part c).

(a) The profit function is $B(x) = (13-\beta x)x - 16 - x - x^2$. Then $B'(x) = 13 - 2\beta x - 1 - 2x = 12 - 2(\beta+1)x$ and $B''(x) = -2(\beta+1) < 0$, i.e., $B(x)$ is a concave function.
Then, $x^* = 3$ solves $B'(x) = 0$, i.e., $12 - 2(\beta+1)3 = 0 \Longrightarrow \beta = 1$.

(b) As $C(x) = 16 + x + x^2$, the average cost is $AC(x) = \frac{16}{x} + 1 + x$, with $AC'(x) = -\frac{16}{x^2} + 1$ and $AC''(x) = \frac{32}{x^3} > 0$, i.e., AC is a convex function.
Then $\widetilde{x}$ such that $AC'(\widetilde{x}) = 0$ minimizes firm's average cost: $\widetilde{x} = 4 > x^* = 3$

(c) Substituting $x^* = 3$ and $\beta = 1$ into profits, we get $B^* = 2$. Substituting $\widetilde{x} = 4$ and $\beta = 1$ into profits, we get $\widetilde{B} = 0$.
So the compensation that the firm will demand will be, at least, 2 monetary units.

(2) **Given the implicit function $y = f(x)$, defined by the equation $2xy - e^y + x^2 = 0$ in a neighbourhood of the point $x = 1, y = 0$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function f at $a = 1$.

(b) approximately sketch the graph of the function f near the point $x = 1$.

(c) approximately sketch the graph of the inverse function of f.

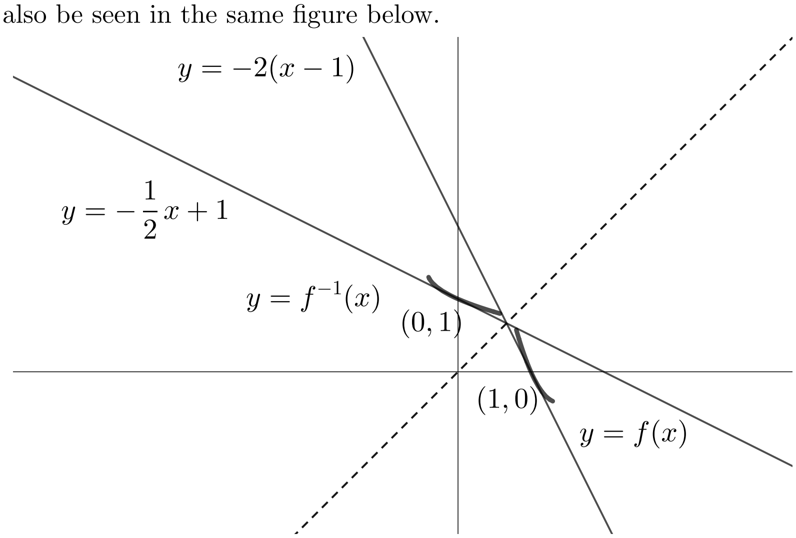
(*Hint for part (b) and (c): use* $f'(1) < 0$, $f''(1) > 0$).

0.4 points part a); 0.2 points part b); 0.4 points part c).

(a) First of all, we notice that $(1, 0)$ is a solution of the equation and the first-order derivative of the equation with respect to the implicit variable y: $2x - e^y$, at the point $x = 1, y = 0$ satisfies the condition $2 - 1 \neq 0$, so the equation can define an implicit function $y = f(x)$ near the point $x = 1, y = 0$. Secondly, we calculate the first-order derivative of the equation: $2y + 2xy' - y'e^y + 2x = 0$ evaluating at $x = 1,\ y(1) = 0$ we obtain: $y'(1) = f'(1) = -2$.
Then the equation of the tangent line is: $y = P_1(x) = 0 - 2(x - 1)$ or $y = -2x + 2$.
Analogously, we calculate the second-order derivative of the equation: $2y' + 2y' + 2xy'' - y''e^y - (y')^2e^y + 2 = 0$ evaluating at $x = 1,\ y(1) = 0,\ y'(1) = -2$ we obtain: $y''(1) = f''(1) = 10$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 0 - 2(x - 1) + 5(x - 1)^2 = -2(x - 1) + 5(x - 1)^2$

(b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 1$, will be as you can see in the figure underneath.

(c) The graph of the inverse function $f^{-1}(x)$, will exist in a neighbourhood of the point $(0, 1)$.
Using symmetry with respect to the principal diagonal ($y = x$), the tangent line to the inverse function at $(0, 1)$ has slope $-\frac{1}{2}$ and its equation is $y = -\frac{1}{2}x + 1$. Therefore, the approximate graph of $f^{-1}(x)$ can also be seen in the same figure below.



(3) **Consider the function $f(x) = \frac{\ln x}{\sqrt[3]{x}}$, defined on the interval $(0, \infty)$. Then:**

(a) find its asymptotes, the intervals where the function $f(x)$ is increasing/decreasing, and its global extreme points.

(b) find the range and sketch the graph of the function.

(c) state the Weierstrass' theorem. Now, consider the new function $f_b(x)$ as $f(x)$ restricted on the interval $[b, \infty)$, where $b > 0$. Discuss for which values of b the thesis (or conclusion) of the Weierstrass' theorem is satisfied.

0.4 points part a); 0.3 points part b); 0.3 points part c)

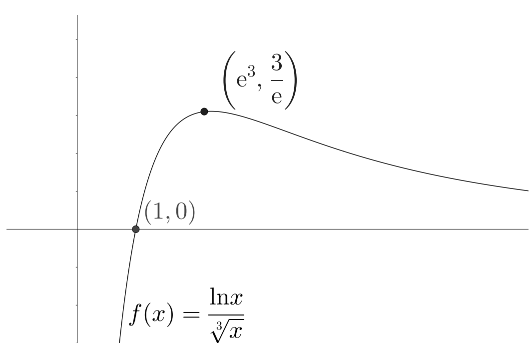
(a) First of all, since $f(x)$ is continuous in its domain we only need to look for asymptotes at 0 and ∞.
$\lim_{x\to 0^+} f(x) = \frac{-\infty}{0^+} = -\infty$, then $f(x)$ has a vertical asymptote at $x = 0$.
$\lim_{x\to\infty} f(x) = \frac{\infty}{\infty} =$ (using L'Hopital) $= \lim_{x\to\infty} \frac{1/x}{x^{-2/3}/3} = \lim_{x\to\infty} \frac{3}{x^{1/3}} = 0$
then $f(x)$ has a horizontal asymptote $y = 0$ at ∞.
Since,

$$f'(x) = \frac{\frac{1}{x}\cdot x^{1/3} - (\ln x)x^{-2/3}/3}{x^{2/3}} = \frac{3x^{-2/3} - (\ln x)x^{-2/3}}{3x^{2/3}} = \frac{3 - \ln x}{3x^{4/3}},$$

we know that $x = e^3$ is the unique critical point.
Calculating $f'(1) > 0$, then $f'(x) > 0$ if $x \in (0, e^3)$, and $f(x)$ is increasing on $(0, e^3]$.
Calculating $f'(e^4) < 0$, then $f'(x) < 0$ if $x \in (e^3, \infty)$, then $f(x)$ is decreasing on $[e^3, \infty)$.
Obviously, $x = e^3$ is the global maximizer of $f(x)$ and $f(x)$ has not global minimizer or minimum point.

(b) Based on the above, the maximum value of the function is $f(e^3) = \frac{3}{e}$ and since $\lim_{x\longrightarrow 0^+} f(x) = -\infty$, using the Intermediate Value theorem for continuous functions, we can deduce that the range of $f(x)$ is $(-\infty, \frac{3}{e}]$.
Therefore, The graph of $f(x)$ will have an appearance, approximately, similar to the one in the figure underneath.

(c) We have seen that $f(x)$ is increasing on $(0, e^3]$, decreasing on $[e^3, \infty)$ and also, $\lim_{x\to\infty} f(x) = 0^+$.
Then, whenever $y = 0$ belongs to the range of $f_b(x)$ the conclusion of the Weierstrass' theorem will be satisfied. Bearing in mind that $f(1) = 0$, we obtain two cases to discuss:
i) if $b \le 1 \Longrightarrow \min(f_b) = f(b)$, $\max(f_b) = \frac{3}{e}$ then the conclusion of the Weierstrass' theorem will be satisfied.
ii) if $b > 1 \Longrightarrow \min(f_b)$ does not exist, and the conclusion of the theorem is not satisfied.
Again, have a Look at the graph of the function.



(4) Let $f(x) = \begin{cases} e^{a(x-1)} & , x \le 1 \\ \frac{b}{2x} & , x > 1 \end{cases}$ **be a piecewise-defined function on** $\mathbb{R}$ **where** $a < 0, b > 0$**, it is asked:**

(a) state the Mean Value theorem (or Lagrange) for a function defined on $[0, 2]$.

(b) find the values of a, b for the function f, so the hypothesis or initial conditions of the theorem are satisfied on $[0, 2]$.

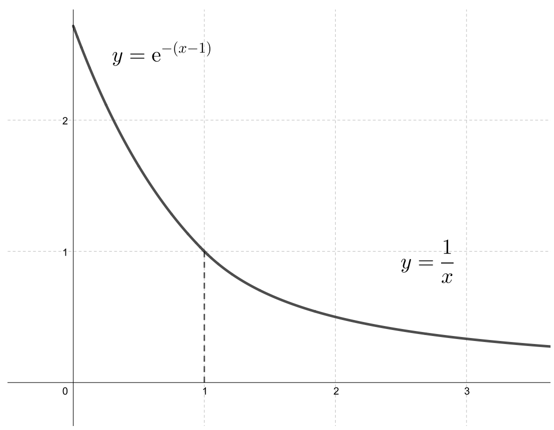
(c) suppose that $a = -\ln 2$ and $b = 2$. Is the thesis or conclusion of the theorem satisfied for the function f on $[0, 2]$?

(*Hint for part c: In order to find the number or point c of the conclusion, start finding it in the interval* $(1, 2)$*).*

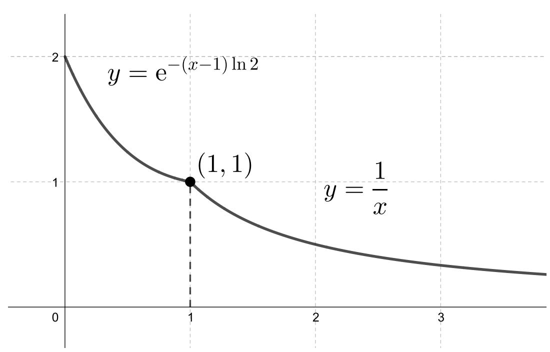
0.2 points part a); 0.6 points part b); 0.2 points part c)

(a) The hypothesis are f is continuous on $[0, 2]$ and derivable on $(0, 2)$.
The thesis or conclusion is that there exists a point $c \in (0, 2)$ such that $f'(c) = (f(2) - f(0))/2$.

(b) First of all, we need that $f(x)$ is continuous on $x = 1$. $\lim_{x\to -1+} f(x) = b/2$; $f(1) = \lim_{x\to -1-} f(x) = 1$ so, $f(x)$ is continuous at $x = 1$ if $b = 2$.
Secondly, supposing f continuous at $x = 1$, the function will be derivable at $x = 1$ when: $f'(1^+) = f'(1^-)$. Then, we obtain:
i)$f'(1^+) = \lim_{x\to 1+} f'(x) = \lim_{x\to 1+} \frac{-b}{2x^2} = \frac{-b}{2} = -1$;
ii) $f'(1^-) = a$, since $f'(x) = ae^{a(x-1)}$.
Finally, the Lagrange's theorem is satisfied when: $b = 2$, $a = -1$.



(c) The thesis or conclusion is that there is a number $c \in (0, 2)$ such that $2f'(c) = f(2) - f(0)$, this is:
i) if $c > 1$, $-2/c^2 = 1/2 - e^{\ln 2} = 1/2 - 2 = -3/2$ then $c^2 = 4/3 > 1 \Longrightarrow c = \frac{2\sqrt{3}}{3} > 1$, and the thesis of the theorem is satisfied.
ii) meanwhile, for the case $c \le 1$ there is no need to be studied.



(5) **Given the functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$, defined by: $f(x) = \ln(2-x)$, $g(x) = 1 + e^{-2x}$, then:**

(a) sketch the set of points A delimited by the graph of the functions $f(x), g(x)$ and the vertical straight lines $x = 0, x = 1$. Find, if they exist, maximal and minimal elements, the maximum and the minimum of A.

(b) calculate the area of the given set.

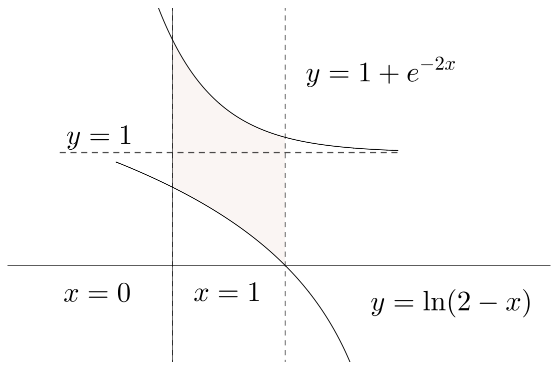
(*Hint for part (a): Pareto's order is defined:* $(x_0, y_0) \leq_P (x_1, y_1) \Longleftrightarrow x_0 \leq x_1, y_0 \leq y_1$).

0.4 points part a); 0.6 points part b)

(a) First of all, we can observe that both functions are positive and increasing on $[0,1]$, and $f(x) < 1 < g(x)$ on the interval.

Furthermore, the line $y = 0$ intersect the graph of $f(x)$ when $x = 1$.

Therefore, the draw of A will be approximately like,



Then, Pareto order describes the set properties:

maximum(A) does not exist, maximal elements$(A) = \{(x, g(x)) : 0 \leq x \leq 1\}$, minimum$(A)$ does not exist and minimal elements$(A) = \{(x, f(x)) : 0 \leq x \leq 1\}$.

(b) First of all, looking at the position of the graphs we know that:

area(A)$=\int_0^1 (g(x) - f(x))dx$.

Integrating by parts, $\int f(x)dx = \int 1 \cdot f(x)dx = x\ln(2-x) - \int x\frac{(-1)}{2-x}dx = x\ln(2-x) + \int \frac{x-2+2}{2-x}dx = x\ln(2-x) + \int(-1 + (-2)\frac{(-1)}{2-x})dx = x\ln(2-x) - x - 2\ln(2-x)$

then applying Barrow's Rule we obtain:

$\int_0^1 f(x)dx = [(x-2)\ln(2-x) - x]_0^1 = -1 - (-2\ln 2) = -1 + 2\ln 2.$

Secondly, since $\int g(x)dx = \int(1 + e^{-2x})dx = x - \frac{1}{2}\int(-2)e^{-2x}dx = x - \frac{1}{2}e^{-2x}$,

then applying Barrow's Rule we obtain:

$\int_0^1 g(x)dx = [x - \frac{1}{2}e^{-2x}]_0^1 = 1 - \frac{1}{2}e^{-2} - (-\frac{1}{2}) = -\frac{1}{2}e^{-2} + \frac{3}{2}.$

Therefore,

area(A)$=\int_0^1 g(x)dx - \int_0^1 f(x)dx = -\frac{1}{2}e^{-2} + \frac{3}{2} - (-1 + 2\ln 2) =$

$= -\frac{1}{2}e^{-2} + \frac{5}{2} - 2\ln 2$ area units.