WORKSHEET 2: Limits and Continuity

1. (*)Calculate

a) $\lim\limits_{x\to 0} \dfrac{4x^3+2x^2-x}{5x^2+2x}$ b) $\lim\limits_{x\to 2} \dfrac{x^3-x^2-x-2}{x-2}$ c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

d) $\lim\limits_{x\to\infty} \dfrac{x^2-\sqrt{x}}{\sqrt{x^3+3x^4}}$ e) $\lim\limits_{x\to\infty} \dfrac{sinx}{x}$ f) $\lim\limits_{x\to-\infty} \dfrac{x^2cosx+1}{x^2+1}$

g) $\lim\limits_{x\to-\infty} \dfrac{3x^3+2x^2+x+2}{x^2-7x+1}$ h) $\lim\limits_{x\to-\infty} \dfrac{x^4-ax^3}{x^2+1}$ i) $\lim\limits_{x\to 0} \dfrac{x^4-x^3}{x^2+b}$

a) $\lim\limits_{x\to 0} \dfrac{4x^3+2x^2-x}{5x^2+2x} = -1/2.$

b) $\lim\limits_{x\to 2} \dfrac{x^3-x^2-x-2}{x-2} = 7$

c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x} = \frac{1}{2\sqrt{2}}$

d) $\lim\limits_{x\to\infty} \dfrac{x^2-\sqrt{x}}{\sqrt{x^3+3x^4}} = \dfrac{1}{\sqrt{3}}$

e) $\lim\limits_{x\to\infty} \dfrac{sinx}{x} = 0.$

f) $\lim\limits_{x\to-\infty} \dfrac{x^2cosx+1}{x^2+1}$ does not exist.

g) $\lim\limits_{x\to-\infty} \dfrac{3x^3+2x^2+x+2}{x^2-7x+1} = -\infty.$

h) $\lim\limits_{x\to-\infty} \dfrac{x^4-ax^3}{x^2+1} = \infty.$

i) $\lim\limits_{x\to 0} \dfrac{x^4-x^3}{x^2+b} = 0.$

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2. Using that $\lim\limits_{x\to 0} \dfrac{sinx}{x} = 1$, calculate:

a)$\lim\limits_{x\to 0} \dfrac{sin^2(2x)}{x^2}$ b)$\lim\limits_{x\to 1} \dfrac{sin(x^2-1)}{x-1}$

a) $\lim\limits_{x\to 0} \dfrac{sin^2(2x)}{x^2} = 4$

b) $\lim\limits_{x\to 1} \dfrac{sin(x^2-1)}{x-1} = 2$

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3. Find the discontinuities,(if they exist) of the following functions:

a)(*)$f(x) = \dfrac{|x-3|}{x-3}$

b)$f(x) = \begin{cases} x+\pi & if \quad x \le -\dfrac{\pi}{2} \\ \dfrac{xsinx}{1-cosx} & if \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2};\quad x \neq 0 \\ 1 & if \quad x = 0 \\ 0 & if \quad \dfrac{\pi}{2} \le x \end{cases}$

c) $f(x) = \begin{cases} \dfrac{x+1}{-x} & if \quad x \le -1. \\ -1/2(1-x^{-2}) & if \quad -1 < x \le 1 \\ \dfrac{sin\pi x}{\pi} - 1 & if \quad 1 < x \end{cases}$

d) (*)$f(x) = \begin{cases} \dfrac{2x}{x+1} & if \quad x < -1. \\ e^{1/x} & if \quad -1 \le x < 0 \\ \pi & if \quad x = 0 \\ 1/x & if \quad 0 < x \end{cases}$

a) f is discontinuous in x=3.

b) f is continuous in $-\pi/2$. On the other hand, f is not continuous in x=0.

Finally, f is not continuous in $\pi/2$.

c) f is continuous in -1. On the other hand, f is not continuous in 1.

d) f is not continuous in -1. On the other hand, f is not continuous in 0.

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4. (*)Calcute the following limits:

   i) $\lim_{x\to 1}\left\{(x-1)arcsin(\frac{tg^4(x)}{1+tg^4(x)})\right\}$

   ii) $\lim_{x\to 2}\frac{1+h^2(x)}{|x-2|}$, with $h(x)$ a function with finite limit when $x\to 2$.

   i) $\lim_{x\to 1}\left\{(x-1)arcsin(\frac{tg^4(x)}{1+tg^4(x)})\right\}=0.$

   ii) $\lim_{x\to 2}\frac{1+h^2(x)}{|x-2|}=\infty.$

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5. (*)Calculate

   a) $\lim_{x\to -3^+}\frac{x^2}{x^2-9}$ b) $\lim_{x\to -3^-}\frac{x^2}{x^2-9}$ c) $\lim_{x\to 0^+}\frac{2}{sinx}$ d) $\lim_{x\to 0^-}(1-1/x)^{\frac{1}{x}}$ e) $\lim_{x\to 0^-}\frac{x^2-2x}{x^3}$

   a) $\lim_{x\to -3^+}\frac{x^2}{x^2-9}=-\infty.$

   b) $\lim_{x\to -3^-}\frac{x^2}{x^2-9}=\infty.$

   c) $\lim_{x\to 0^+}\frac{2}{sinx}=\infty.$

   d) $\lim_{x\to 0^-}(1-1/x)^{\frac{1}{x}}=0.$

   e) $\lim_{x\to 0^-}\frac{x^2-2x}{x^3}=-\infty$

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6. Calculate all asymptotes of the following functions:

   $(*)f(x)=\frac{x^3}{x^2-1}$ $g(x)=\frac{x^2-1}{x}$ $(*)h(x)=\sqrt{x^2-1}$ $(*)m(x)=\frac{1}{lnx}$ $(*)n(x)=e^{1/x}$

   a) $y=x$ is a oblique asymptote in ∞ (and, analogously, in $-\infty$). On the other hand,$x=1$, $x=-1$ are the vertical asymptotes.

b) $y=x$ is a oblique asymptote in ∞ (and analogously, in $-\infty$). On the other hand, $x=0$ is the only vertical asymptote.
c) $y=x$ is a oblique asymptote in ∞.However, the asymptote in $-\infty$ is $y=-x$.
d) $y=0$ is the horizontal asymptote in ∞, $x=0$ is not a vertical asymptote, $x=1$ is a vertical asymptote.
e) $y=1$ is the horizontal asymptote in ∞ and in -∞, $x=0$ is the only vertical asymptote.
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7. Prove that every odd-degree polynomial has at least one root.

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8. (*)a) Use the intermediate value theorem to check that the following functions have a zero at the specified interval

   i)$f(x)=x^2-4x+3$ in $[2,4]$; ii)$g(x)=x^3+3x-2$ in $[0,1]$.

   b) Obtain using interval partitions and succesive applications of Bolzano, the zero with and error of ± 0.25.

   x=3 is a zero of f with total accuracy. On the other hand, x=3/4 is a zero of g with an error less than ± 0.25.

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9. (*)Check that the equations $x^4 - 11x + 7 = 0$ and $2^x - 4x = 0$ have at least two solutions.

   a) There is a root between 0 and 1, and another root between 1 and 2.

   b) There is a root between 0 and 1. On the other hand $g(4) = 0$.

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10. (*)Prove that the equation $x^7 + 3x + 3 = 0$ has a unique solution. Determine the integer part of that solution.

    The integer part of the solution is -1.

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11. Find the domain and the range of the functions:

    a)$f(x) = ln\left(\frac{(x^2-16)(x-1)}{x-3}\right)$ b)$g(x) = \sqrt{\frac{(x^2-16)(x-1)}{x-3}}$

    a) $Dom(f) = (-\infty, -4) \cup (1, 3) \cup (4, \infty)$; $Range(f) = \mathbb{R}$.

    b) $Dom(g) = (-\infty, -4] \cup [1, 3) \cup [4, \infty)$; $Range(g) = [0, \infty)$.

12. If f and g are continous functions in $[a, b]$ and $f(a) < g(a)$, $f(b) > g(b)$, prove that there exists a $x_0 \in (a, b)$ such that $f(x_0) = g(x_0)$

13. a) Let $f : [a, b] \to \mathbb{R}$, continuous, such that Range$(f) \subset [a, b]$. Prove that f has at least a fixed point.

    b) Also suppose that f is monotonic. Will exist an unique fixed point?

14. a) Prove using the Bolzano's theorem of zeroes, that the function $f(x) = x^3 - 5$ has at least one fixed point in the interval $[0, n]$, for some $n \in \mathbb{N}$.

    b) Obtain, with an error of ± 0.25, a fixed point of f.
    c) Does a unique fixed point exist?
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    a) $f(x) = x^3 - 5$ has a fixed point in $(0, 2)$.
    b) f has a fixed point in $\frac{7}{4}$ with an error less than ± 0.25.
    c) The fixed point will be unique in any interval $[0, n]$.
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15. (*)Discuss in the following cases if the functions reach global and/or local extrema in the specified intervals:

    a) $f(x) = x^2 \quad x \in [-1, 1]$ b) $f(x) = x^3 \quad x \in [-1, 1]$

    c) $f(x) = sinx \quad x \in [0, \pi]$ d) $f(x) = -x^{\frac{1}{3}} \quad x \in [-1, 1]$
    a) f reaches global maximun in -1 and in 1. It does not reach local maxima.
    f reaches local and global minumun in 0.
    b) f reaches global minimun in -1 and global maximun in 1.
    It does not reach local extrema.
    c) f reaches local and global maximun in $\pi/2$, and global minima in 0 and in π.
    It does not reach local minima.
    d) f reaches global minimun in 1 and global maximun in -1.
    It does not reach local extrema.
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16. In the previous problem, replace the interval given by $[0, \infty)$ or by $\mathbb{R}$ in each one of the functions.

17. Let $f(x) = arctg\left(\frac{tg^2x}{1 + tg^4x}\right)$, $f : [a, b] \to \mathbb{R}$. Discuss, depending on the values of a and b, when f reaches maximun and minimun in $[a, b]$.

18. Explain why $f(x) = tgx$ has a maximun in $[0, \pi/4]$, but not in $[0, \pi]$.

19. (*)a) Let $C(x) = \frac{3x^2 + x}{x - 1} + 100$, be the total cost of production function, supposing $x \geq 7$.

    Check if it has oblique asymptote when $x \to \infty$.

b) Consider the function $C_m(x) = \dfrac{C(x)}{x}$, that is, the average cost of production.
Check that it has a horizontal asymptothe when $x \to \infty$.
c) Is there any relationship between the oblique asymptote in part a) and the horizontal asymptote in part b?

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a) $y = 3x + 104$ is the oblique asymptote in ∞ of $C(x)$.
b) Obviously, is $y = 3$.
c) In effect, the coefficient of the x in the oblique asymptote in part a) is the constant term in part b).
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20. (*)A banking entity offers a current account with the following conditions: the 250.000 fist euros non remunerated, the rest by a 7% of annual interest. Consider the following function: $i : [0, \infty) \to \mathbb{R}$ defined by $i(x)$="interes obtained in % when depositing some capital x and mantaining it during a year".

   i) Obtain $i(x)$.

   ii) Calculate $\lim\limits_{x \to \infty} i(x)$.

   iii) Does any capital c exist such that $i(c) = 7$?.

   iv) From what capital is obtained at least a 6% of interest?

   v) Graph the function i.

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a) $i(x) = 7(x - 250.000)/x$, if $x \geqslant 250.000; 0$, if $x < 250.000$.
b) 7.
c) No.
d) when $x = 1.750.000$