Chapter 5

Integration

5.1 The indefinite integral

In many respects, the operation of integration that we are studying here is the inverse operation of derivation.

Definition 5.1.1. The function F is an antiderivative (or primitive) of the function f on the interval I if $F'(x) = f(x)$ for all $x \in I$.

Thus, both $F_1(x) = x^3 + 6$ and $F_2(x) = x^3 - 2$ are antiderivatives of $f(x) = 3x^2$ in any interval.

Theorem 5.1.2. *If F_1 and F_2 are two arbitrary antiderivatives of f on I, then $F_1(x) - F_2(x) = const.$ on I.*

Proof. By definition of antiderivative $F_1' = F_2' = f$ on I, thus $(F_1 - F_2)'(x) = 0$ for every $x \in I$. Since a function with a null derivative on an interval is constant, we have $F_1(x) - F_2(x) = \text{const}$. □

Corollary 5.1.3. *If F is one of the antiderivatives of f on I, any other antiderivative G of the function f on I has the form $G(x) = F(x) + C$, where C is a constant.*

Definition 5.1.4. The set of all antiderivatives of the function f on the interval I is called the indefinite integral of f on I, and it is denoted

$$\int f(x)\, dx.$$

Notice that by Corollary **??**, $\int f(x)\, dx = F(x) + C$, where F is one of the antiderivatives of f on I, and C is an arbitrary constant. Sometimes the symbol $\int f(x)\, dx$ denotes not the whole set of antiderivatives but any one of them.

5.1.1 Properties of the Indefinite Integral

1. $\int F'(x)\, dx = F(x) + C$;

2. For any functions f, g and constants a, b, $\int (af(x) + bg(x))\, dx = a\int f(x)\, dx + b\int g(x)\, dx$.

5.1.2 Basic Indefinite Integrals

1. $\displaystyle\int 0\,dx = C$;

2. $\displaystyle\int 1\,dx = x + C$;

3. $\displaystyle\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \qquad (a \neq -1)$;

4. $\displaystyle\int \frac{dx}{x} = \ln|x| + C \qquad (x \neq 0)$;

5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (0 < a \neq 1),\ \int e^x\,dx = e^x + C$;

6. $\displaystyle\int \sin x\,dx = -\cos x + C$;

7. $\displaystyle\int \cos x\,dx = \sin x + C$;

8. $\displaystyle\int \frac{1}{\cos^2 x}\,dx = \tan x + C \qquad (x \neq \frac{\pi}{2} + k\pi,\ k \text{ integer})$;

9. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \qquad (-1 < x < 1)$;

10. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$.

5.1.3 Integration by Change of Variable

Sometimes the task of finding the integral $\int f(x)\,dx$ is simplified by means of a change of variable $x = \varphi(t)$. The *formula of change of variable* in an indefinite integral is

$$\int f(x)\,dx\Big|_{x=\varphi(t)} = \int f(\varphi(t))\varphi'(t)\,dt.$$

Example 5.1.5. Find $\int \tan x\,dx$.

Solution: Let $t = \cos x$. Then $dt = -\sin x\,dx$. Hence, by the formula of change of variable

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{dt}{t} = -\ln|t| + C = -\ln|\cos x| + C.$$

Example 5.1.6. Find $\int \sqrt{2x-1}\,dx$.

Solution: Let $t = 2x - 1$. Then $dt = 2dx$. Hence,

$$\int \sqrt{2x-1}\,dx = \frac{1}{2}\int \sqrt{t}\,dt = \frac{1}{2}\int t^{1/2}\,dt = \frac{1}{2}\frac{t^{3/2}}{3/2} + C = \frac{1}{3}t^{3/2} + C = \frac{1}{3}(2x-1)^{3/2} + C.$$

Example 5.1.7. Find $\int x\sqrt{2x-1}\,dx$.

Solution: Let $t = 2x - 1$. Then $dt = 2dx$. Moreover, $x = (1+t)/2$. Applying the change of variable formula we get

$$\int x\sqrt{2x-1}\,dx = \frac{1}{4}\int (1+t)t^{1/2}\,dt = \frac{1}{4}\int t^{1/2} + t^{3/2}\,dt = \frac{1}{4}\left(\frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2}\right) + C$$

$$= \frac{1}{4}[\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}] + C = \frac{1}{6}t^{3/2} + \frac{1}{10}t^{5/2} + C = \frac{1}{6}(2x-1)^{3/2} + \frac{1}{10}(2x-1)^{5/2} + C$$

Example 5.1.8. Find $\int \frac{\ln x}{x}\,dx$.

Solution: Let $t = \ln x$. Then $dt = dx/x$ and

$$\int \frac{\ln x}{x}\,dx = \int t\,dt = \frac{1}{2}t^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

Example 5.1.9. Find $\int xe^{-x^2}\,dx$.

Solution: Let $t = x^2$. Then $dt = 2xdx$ and

$$\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-t}\,dt = -\frac{1}{2}e^{-t} + C = -\frac{1}{2}e^{-x^2} + C.$$

5.1.4 Integration by parts

For differentiable functions u and v we have $(uv)' = uv' + vu'$. Taking integrals and given that $\int (uv)'(x)\,dx = u(x)v(x)$, we have

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$$

This relation is known as the *formula of integration by parts*. Using the identifications $u'(x)\,dx = du$ and $v'(x)\,dx = dv$ we can write this formula as

$$\int u\,dv = uv - \int v\,du.$$

Example 5.1.10. Find $\int xe^x\,dx$.

Solution: Let $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$. Hence

$$\int xe^x\,dx = xe^x - \int e^x\,dx = e^x(x-1) + C.$$

Example 5.1.11. Find $\int x^2 \ln x \, dx$.

SOLUTION: Let $u = \ln x$ and $dv = x^2 \, dx$. Notice that $du = dx/x$ and $v = x^3/3$. Then, using the formula of integration by parts we get

$$\int x^2 \ln x \, dx = \ln x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3x} \, dx = \ln x \left(\frac{x^3}{3}\right) - \frac{1}{3} \int x^2 \, dx = \ln x \left(\frac{x^3}{3}\right) - \frac{1}{9} x^3 + C.$$

Example 5.1.12. Find $\int \arctan x \, dx$.

SOLUTION: Let $u = \arctan x$ and $dv = dx$. Then $du = dx/(1+x^2)$ and $v = x$. Hence

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx.$$

Now, observe that using the change of variable $t = x^2$ we have $dt = 2x \, dx$ thereby

$$\int \frac{x}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{1+t} \, dt = \frac{1}{2} \ln |1+t| + C = \frac{1}{2} \ln (1+x^2) + C.$$

Plugging this value into the above expression we finally get

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln (1+x^2) + C.$$

Example 5.1.13. Find $\int x^2 \sin x \, dx$.

SOLUTION: Let $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x$. Thus

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx.$$

Applying again integration by parts to the second integral, $u = x$ and $dv = \cos x \, dx$ we have $du = dx$ and $v = \sin x$, hence

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Plugging this value into the previous expression we finally get

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

5.1.5 Integration of Rational Functions

A rational function is of the form $\frac{P_n(x)}{Q_m(x)}$, where P_n and Q_m are polynomials of degrees n and m, respectively. If $n \geq m$ the fraction is *improper* and can be represented

$$\frac{P_n(x)}{Q_m(x)} = P_{n-m}(x) + \frac{R_k(x)}{Q_m(x)},$$

where the degree of the polynomial R_k is $k < m$. Thus, the integration of an improper fraction can be reduced to the integration of a *proper* fraction ($k < m$)

$$\int \frac{P_n(x)}{Q_m(x)}\,dx = \int P_{n-m}(x)\,dx + \int \frac{R_k(x)}{Q_m(x)}\,dx.$$

Example 5.1.14.

$$\int \frac{x^3+x^2+x}{x^2+1}\,dx = \int (x+1)\,dx - \int \frac{1}{x^2+1}\,dx,$$

because dividing the polynomials we find

$$\frac{x^3+x^2+x}{x^2+1} = x+1-\frac{1}{x^2+1}.$$

Then

$$\int \frac{x^3+x^2+x}{x^2+1}\,dx = \frac{1}{2}(x+1)^2 - \arctan x + C.$$

Theorem 5.1.15. *Suppose that* $\dfrac{P_n(x)}{Q_m(x)}$ *is a proper fraction* ($n < m$) *and that*

$$Q_m(x) = (x-a)^\alpha \cdots (x-b)^\beta,$$

where $a, \ldots, b$ *are real roots of multiplicity* $\alpha, \ldots, \beta$. *Then there are constants* $A_i \ldots B_i$, *such that*

$$\begin{aligned}\frac{P_n(x)}{Q_m(x)} &= \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \cdots + \frac{A_1}{x-a}\\ &\quad + \cdots + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \cdots + \frac{B_1}{x-b}\end{aligned}$$

Example 5.1.16. An important consequence is that for a proper fraction that satisfies the condition $\alpha = \cdots = \beta = 1$, we have

$$\begin{aligned}\int \frac{P_n(x)}{Q_m(x)}\,dx &= \int \frac{A}{x-a}\,dx + \cdots + \int \frac{B}{x-b}\,dx\\ &= A\ln|x-a| + \cdots + B\ln|x-b| + C.\end{aligned}$$

The fractions which appear on the right–hand side are partial fractions and the relation is the decomposition of a proper rational fraction into a sum of partial fractions.

Example 5.1.17. Find $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$.

Solution: Notice that $x^2-5x+6 = (x-3)(x-2)$. Then

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2)+B(x-3)}{(x-3)(x-2)}.$$

Setting $x = 2$ we get $1 = -B$, whence $B = -1$ and setting $x = 3$ we get $A = 1$. Hence

$$\int \frac{1}{x^2-5x+6}\,dx = \ln|x-3| - \ln|x-2| + C.$$

Each of the partial fractions can be integrated in terms of elementary functions:

Example 5.1.18.

$\int \frac{A}{(x-a)^\alpha}\, dx = \frac{A}{1-\alpha}\frac{1}{(x-a)^{\alpha-1}} + C, \quad (\alpha > 1)$

Example 5.1.19. If $x^2 + px + q$ has no real roots

$$\int \frac{Mx+N}{x^2+px+q}\, dx = \int \frac{M}{2}\frac{2x+p}{x^2+px+q} + (N - \frac{Mp}{2}) \int \frac{dx}{x^2+px+q}$$

Now computing both integrals separately, we have that for the first term

$$\int \frac{M}{2}\frac{2x+p}{x^2+px+q}\, dx = \frac{M}{2}\ln(x^2+px+q)$$

and for the second term

$$(N - \frac{Mp}{2}) \int \frac{dx}{x^2+px+q} = \frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}} \int \frac{\sqrt{q-\frac{p^2}{4}}}{x^2+px+q} dx =$$

$$\frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}} \int \frac{\frac{1}{\sqrt{q-\frac{p^2}{4}}}}{\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right)^2 + 1} dx = \frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}} \arctan \frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}$$

And finally

$$\int \frac{Mx+N}{x^2+px+q}\, dx = \frac{M}{2}\ln(x^2+px+q) + \frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}} \arctan \frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C$$

5.2 The Definite Integral

Definition 5.2.1. The definite integral of a non–negative continuous function f on the interval $I = [a, b]$ is the area, A, of the region bounded by the graph of f, the horizontal axis, and the two line segments $x = a$, $x = b$. It is denoted

$$\int_a^b f(x)\, dx = A.$$

Example 5.2.2. If $f(x) = 1 - x$, then $\displaystyle\int_0^1 f(x)\, dx = 1/2$, since the figure under the graph of f limited by $x = 0$, $x = 1$ is a square triangle with area $1/2$.

Definition 5.2.3. The definite integral of a non–positive continuous function f on the interval $I = [a, b]$ is the area with negative sign of the region bounded by the graph of f, the horizontal axis, and the two line segments $x = a$, $x = b$. Hence

$$\int_a^b f(x)\, dx = -A.$$

It is straightforward to define the definite integral of a function that change sign in the interval $[a, b]$. By way of example, suppose that f is continuous on $[a, b]$ and satisfies $f \geq 0$ in $[a, c]$, $f \leq 0$ in $[c, b]$. Then the definite integral of f on $[a, b]$ is the difference of the areas

$$\int_a^b f(x)\,dx = A_{[a,c]} - A_{[c,b]} = \int_a^c f(x)\,dx - \int_c^b (-f)(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

(see Property (4) below).

More complex situations can be similarly handled.

Example 5.2.4 (Example **??**, continued)**.** If $f(x) = 1 - x$, then $\int_0^2 f(x)\,dx = 0$, since we know $\int_0^1 f(x)\,dx = 1/2$ and $\int_1^2 (-f)(x)\,dx = -1/2$. The latter is because the region bounded by $-f$ between $x = 1$ and $x = 2$ is again a square triangle with area $1/2$.

5.2.1 Properties of the definite integral

In what follows f and g are continuous functions on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$.

1. $\displaystyle\int_a^a f(x)\,dx = 0;$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx;$
3. $\displaystyle\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$
4. For any $c \in [a, b]$, $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$
5. If $f(x) \geq g(x)$ in $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$
6. If $f(x)$ is an increasing function on $[a, b]$, then $f(a) \leq \int_a^b f(x)dx \leq f(b)$.
7. If $f(x)$ is a decreasing function on $[a, b]$, then $f(b) \leq \int_a^b f(x)dx \leq f(a)$.
8. If $f(x)$ is a convex function on $[a, b]$, then $\int_a^b f(x)dx \leq [f(a) + f(b)](b - a)/2$.
9. If $f(x)$ is a concave function on $[a, b]$, then $[f(a) + f(b)](b - a)/2 \leq \int_a^b f(x)dx$.

5.3 Barrow's Rule

In this section we show the connection between areas and antiderivatives.

Definition 5.3.1. Let the function f be continuous on the interval $[a, b]$. The function

$$F(x) = \int_a^x f(t)\,dt \qquad (a \leq x \leq b)$$

is called an integral with a variable upper limit.

Theorem 5.3.2 (Fundamental Theorem of Integral Calculus)**.** *If the function f is continuous on the interval $[a, b]$, then the function $F(x) = \displaystyle\int_a^x f(t)\,dt$ is an antiderivative of f in $[a, b]$.*

Written in other terms, the theorem establishes

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

Theorem 5.3.3 (Barrow's Rule)**.** *If the function f is continuous on the interval $[a, b]$, then*

$$\int_a^b f(x)\,dx = G(b) - G(a),$$

holds true, where G is an antiderivative of f in $[a, b]$.

Proof. Let G be an arbitrary antiderivative of f in $[a, b]$. Then $G - F$ is constant in $[a, b]$ by Theorem **??**, where $F(x) = \int_a^x f(t)\,dt$, because F is also an antiderivative of f. Hence $G(a) - F(a) = G(b) - F(b)$, or

$$G(b) - G(a) = F(b) - F(a) = \int_a^b f(x)\,dx - \int_a^a f(x)\,dx = \int_a^b f(x)\,dx.$$

□

Most often we will write $G(b) - G(a)$ as $G(x)|_a^b$.

Theorem 5.3.4 (Change of variable)**.** *Let f be continuous in $[a, b]$, and let $x = g(t)$ be differentiable and increasing in $[\alpha, \beta]$, where $g(\alpha) = a$, $g(\beta) = b$ and $a \leq g(t) \leq b$. Then*

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))g'(t)\,dt.$$

Theorem 5.3.5 (Integration by parts)**.** *If f and g have continuous derivatives in $[a, b]$, then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

5.3.1 The area of a plane figure

Given a continuous function f, the area of the figure bounded by the curve $y = f(x)$, the axis OX and the line segments $x = a$, $x = b$ is

$$A = \int_a^b |f(x)|\, dx.$$

Example 5.3.6 (Example **??**, continued)**.** The area of the figure limited by $y = 1 - x$ in the interval $[0, 2]$ is

$$A = \int_0^2 |1 - x|\, dx = \int_0^1 (1 - x)\, dx + \int_1^2 (x - 1)\, dx = \frac{1}{2} + \frac{1}{2} = 1.$$

Example 5.3.7. The area of the figure limited by the graph of $y = \ln x$ and the horizontal axis and the line segments $x = 1/e$, $x = e$ is

$$A = \int_{\frac{1}{e}}^e |\ln x|\, dx.$$

The logarithm is negative in $[1/e, 1]$ and positive in $[1, e]$. thus

$$A = \int_{\frac{1}{e}}^1 (-\ln x)\, dx + \int_1^e \ln x\, dx$$

The integral can be solved using parts $u = \ln x$, $dv = dx$ obtaining

$$\int_{\frac{1}{e}}^1 \ln x\, dx = x \ln x \Big|_{\frac{1}{e}}^1 - x \Big|_{\frac{1}{e}}^1 = -1 + \frac{2}{e},$$

$$\int_1^e \ln x\, dx = x \ln x \Big|_1^e - x \Big|_1^e = 1.$$

Thus, $A = -(-1 + 2/e) + 1 = 2(1 - 1/e)$.

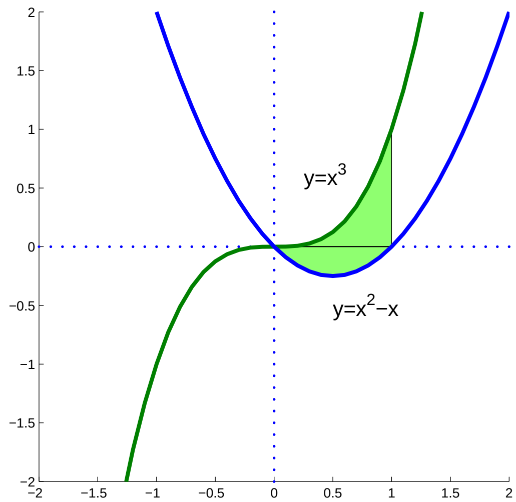
Suppose that a plane figure is bounded by the continuous curves $y = f(x)$, $y = g(x)$, $a \leq x \leq b$, where $g(x) \leq f(x)$, and two line segments $x = a$, $x = b$ (the line segments may degenerate into a point). Then the area of the figure is

$$A = \int_a^b (f(x) - g(x))\, dx.$$

Example 5.3.8. Find the area of the figure bounded by the curves $y = x^3$, $y = x^2 - x$ in the interval $[0, 1]$.

Solution: The curves meet at a single point. Solving the equation $x^3 = x^2 - x$, we find the abscissa of the point, $x = 0$. Hence one of the curves remains above the other in the whole interval. To know which, we simply substitute into $x^3 - x^2 + x$ an arbitrary value in the interval; for $x = 1/2$ we get $x^3 - x^2 + x|_{x=1/2} = 0.375 > 0$, thus x^3 is above $x^2 - x$ in $[0, 1]$. The area is

$$A = \int_0^1 x^3 - (x^2 - x)\, dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1 = \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{2}\right) - 0 = \frac{5}{12}.$$



Example 5.3.9. Find the area of the figure bounded by the curves $y = 2 - x^2$, $y = x$.

SOLUTION: The curves meet at two points. Solving the equation $2 - x^2 = x$ we find that the points are $x = -2$, $x = 1$. Hence one of the curves remains above the other in the interval $[-2, 1]$. To know which, we simply substitute into $2 - x^2 - x$ an arbitrary value on the interval $[-2, 1]$; for $x = 0$, $(2 - x^2 - x)|_{x=0} = 2 > 0$, thus $y = 2 - x^2$ is above $y = x$ in $[-2, 1]$. The area is

$$A = \int_{-2}^{1} (2 - x^2 - x)\, dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) = \frac{9}{2}.$$