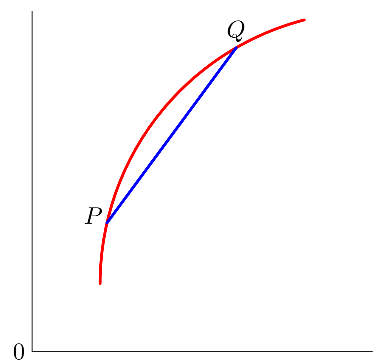
Chapter 3

Derivatives

3.1 Derivative of a function. Differentiation rules

3.1.1 Slope of a curve

The slope of a curve at a point P is a measure of the steepness of the curve. If Q is a point on the curve near P, then the slope of the curve at P is approximately the slope of the line segment $\overline{PQ}$. The *slope of the curve* at P is defined to be the limit of the slope of $\overline{PQ}$ as Q approaches P along the curve.



In symbols, slope of curve at $P = \lim_{Q\to P}(\text{slope of } \overline{PQ})$.

To find the slope of the curve $y = x^2$ at point $P = (1, 1)$ we choose a point Q on the curve near P. Let the x–coordinate of Q be $1 + h$ with h small. The y–coordinate of Q is $(1 + h)^2$. We now calculate

$$\text{slope of } \overline{PQ} = \frac{(1+h)^2 - 1}{(1+h) - 1} = 2 + h.$$

As Q approaches P, h approaches 0. Thus:

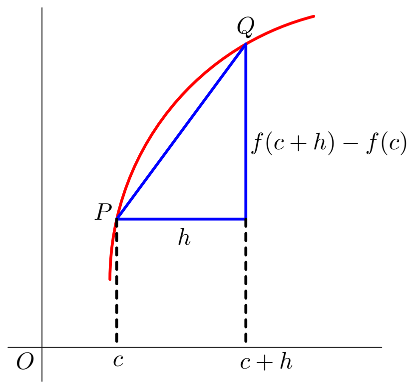
$$\text{slope of curve at } (1, 1) = \lim_{h\to 0}(2 + h) = 2.$$

Definition 3.1.1. The derivative of function f at point c, $f'(c)$, is the slope of the curve $y = f(x)$ at point $(c, f(c))$, that is:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

whenever the limit exists.

We shall say that f is *differentiable* at point c if $f'(c)$ exists.



3.1.2 Table of derivatives of basic elementary functions

1. $(x^\alpha)' = \alpha x^{\alpha-1}$ (α is any number).
2. $(\ln x)' = \frac{1}{x}$.
3. $(a^x)' = a^x \ln a$, in particular $(e^x)' = e^x$.
4. $(\sin x)' = \cos x$.
5. $(\cos x)' = -\sin x$.
6. $(\tan x)' = \dfrac{1}{\cos^2 x}$, $(x \neq \frac{\pi}{2} + \pi n$, n integer).
7. $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$.
8. $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$.
9. $(\arctan x)' = \dfrac{1}{1+x^2}$.

3.1.3 The line tangent to a curve

The line tangent to a curve at a point is defined to be the line that passes through the point and that has slope that is the same as the slope of the curve at that point. Thus,

$$y - f(c) = f'(c)(x - c)$$

is the the equation of the line tangent to $y = f(x)$ at point $P = (c, f(c))$.

Example 3.1.2.

1. Find the line tangent to $y = \sqrt{x}$ at $(16, 4)$.

   SOLUTION: $f(x) = x^{1/2}, \quad f'(x) = \dfrac{1}{2}x^{-1/2}, \quad f'(16) = \dfrac{1}{8}.$
   Hence, $y - 4 = \dfrac{1}{8}(x - 16)$, or $y = \dfrac{1}{8}x + 2$.

2. Find the line tangent to $y = |x|$ at $(0, 0)$.

   SOLUTION: There is no tangent line to $y = |x|$ at $(0, 0)$, since the function $f(x) = |x|$ is not differentiable at 0. To see this, notice that the limit

   $$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

   does not exist (the left limit is -1 and the right limit is 1).

3.1.4 One–sided derivatives

If there is the limit

$$\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \qquad \left(\lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}\right),$$

then it is called the *right–hand* (*left–hand*) *derivative* of the function f at the point c and is denoted $f'(c^+)$ $(f'(c^-))$.

Theorem 3.1.3. *$f'(c)$ exists if and only if both $f'(c^+)$ and $f'(c^-)$ exists and they are equal. In this case, $f'(c) = f'(c^+) = f'(c^-)$.*

Example 3.1.4. Is the function $f(x) = \begin{cases} x^2, & \text{if } x \le 0; \\ xe^{-1/x}, & \text{if } x > 0. \end{cases}$ differentiable at 0?

SOLUTION: Yes, and $f'(0) = 0$.

$$f'(0^-) = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{h^2}{h} = 0;$$
$$f'(0^+) = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{he^{-1/h}}{h} = e^{-\infty} = 0.$$

3.1.5 Continuity and differentiability

Continuity is a necessary condition for differentiability. In other words, a discontinuous function cannot be differentiable.

Theorem 3.1.5. *Let f be differentiable at c. Then, it is continuous at c.*

Proof. By assumption the limit

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists. We want to prove that f is continuous at c, that is, $\lim_{h\to 0} f(c+h) = f(c)$ or, equivalently, that $\lim_{h\to 0} \big(f(c+h) - f(c)\big) = 0$. To this end consider the following computations:

$$\lim_{h\to 0} \frac{h}{h}\big(f(c+h) - f(c)\big) = \lim_{h\to 0} h \cdot \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = 0 \cdot f'(c) = 0.$$

□

Example 3.1.6. Discuss the differentiability of the function $f(x) = \begin{cases} ax - x^2, & \text{if } x < 1; \\ b(x-1), & \text{if } x \geq 1. \end{cases}$, where $a, b \in \mathbb{R}$.

SOLUTION: First we study continuity. The domain of f is the whole real line. For $x < 1$ and $x > 1$ it is given by elementary functions, which are continuous. It remains to consider the frontier point $x = 1$. We have $f(1) = 0$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax - x^2 = a - 1$. Thus, f is continuous at 1 if and only if $a = 1$. Hence

> If $x \neq 1$, then f is continuous for any a, b and at $x = 1$, f is continuous if and only if $a = 1$ (b arbitrary).

Now we go with differentiability. Clearly, f is differentiable at any point $x \neq 1$. When $a \neq 1$ f is not differentiable at $x = 1$ since it is not continuous at this point. Hence, let us consider $a = 1$.

$$\begin{aligned} f'(1^-) &= \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h) - (1+h)^2 - 0}{h} \\ &= \lim_{h\to 0^-} \frac{(1+h) - (1+2h+h^2)}{h} = \lim_{h\to 0^-} \frac{-h-h^2}{h} = -1; \\ f'(1^+) &= \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{b(1+h-1)}{h} = b. \end{aligned}$$

Hence $f'(1^-) = f'(1^+) = f'(1)$ if and only if $b = -1$. In summary

> If $x \neq 1$, then f is differentiable for any a, b and at $x = 1$, f is differentiable if and only if $a = 1$ and $b = -1$.

3.1.6 Rules of differentiation

Let f and g be functions differentiable at point c. Then, the sum, difference, product by a scalar, product and quotient are also differentiable at c and the derivatives are given by the following expressions.

1. Sum: $(f+g)' = f' + g'$;
2. Difference: $(f-g)' = f' - g'$;

3. Product by a scalar: $(\lambda f)' = \lambda f'$, $\lambda \in \mathbb{R}$;
4. Product: $(f \cdot g)' = f' \cdot g + f \cdot g'$;
5. Quotient: $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, $g(c) \neq 0$.

3.1.7 Chain Rule

(Derivative of a compose function). Let f be differentiable at c and let g be differentiable at $f(c)$. Then the composition $g \circ f$ is differentiable at c and the derivative

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Example 3.1.7. Find the derivative of $y = \sin(3^x + x^3)$.

Solution: We can represent the function in the form $y = \sin t$ where $t = 3^x + x^3$. Using the chain rule we get

$$y' = (\sin t)'|_{t=3^x+x^3}(3^x + x^3)' = \cos(3^x + x^3)(3^x \ln 3 + 3x^2).$$

Example 3.1.8. Find the derivative of $h(x) = \sqrt{e^x - x^2}$ at the point $c = 1$.

Solution:

$$h'(x) = \left((e^x - x^2)^{\frac{1}{2}}\right)' = \frac{1}{2}(e^x - x^2)^{-\frac{1}{2}}(e^x - x^2)' = \frac{1}{2}(e^x - x^2)^{-\frac{1}{2}}(e^x - 2x).$$

Hence $h'(1) = \frac{e-2}{2\sqrt{e-1}}$, approximately 0.274

Example 3.1.9. In the following examples it is supposed that f is a differentiable function.

- $\left(e^{f(x)}\right)' = f'(x)e^{f(x)}$;
- $\left(a^{f(x)}\right)' = (\ln a)f'(x)a^{f(x)}$.
- $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$;
- $(\arctan f(x))' = \dfrac{f'(x)}{1 + f^2(x)}$.

3.1.8 Derivative of the inverse function

Let f be continuous and one-to-one in an open interval $(x - \delta, x + \delta)$ and such that $f'(x)$ exists. Then f^{-1} is differentiable at $y = f(x)$ and the derivative is

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}. \tag{3.1.1}$$

The proof of this assertion is easy using the Chain Rule. For, deriving in both sides of the identity

$$x = f^{-1}(f(x)),$$

we obtain

$$1 = \left(f^{-1}\right)'(f(x)) \cdot f'(x).$$

The formula follows, once we substitute $y = f(x)$.

Example 3.1.10. Prove that $\arctan' x = \dfrac{1}{1+x^2}$.

Solution: The function $\arctan x$ is the inverse of the function $\tan x$. According to the formula (3.1.1) above

$$\arctan' y = \frac{1}{1+\tan^2 x},$$

because $\tan' x = 1 + \tan^2 x$, and where $y = \tan x$. Thus,

$$\arctan' y = \frac{1}{1+y^2}.$$

Of course, we can change the name of the variable from y to x to get the result.

3.1.9 Using the derivative for approximations

The line tangent to a curve at a point $(c, f(c))$ coincides with the curve at the point of tangency, and constitute a good approximation of the curve at points near $(c, f(c))$. In fact, a function is differentiable at a point when the graph of the function at this point can be well approximated by a straight line (the tangent line).

Thus, for small values of h, the value of $f(c+h)$ can be approximated by the known valued of $f(c)$ and $f'(c)$:

$$f(c+h) \approx f(c) + f'(c)h. \tag{3.1.2}$$

Another way to express this is by increments : the increase in the value of f, $\triangle y$, is approximately the increment in x, $h = \triangle x$, multiplied by $f\ '(c)$:

$$\triangle y \approx f'(c)\triangle x$$

Example 3.1.11. Without using a pocket calculator, give an approximated value of $\sqrt{0.98}$.

Solution: Let us consider the function $f(x) = \sqrt{1+x}$. Notice that $f(0) = 1$, $f(-0.02) = \sqrt{0.98}$, $f'(x) = \dfrac{1}{2}(1+x)^{-1/2}$, $f'(0) = 0.5$. We find from formula (3.1.2) with $c = 0$ and $h = -0.02$ that

$$\sqrt{0.98} = f(0-0.02) \approx f(0) + f'(0)(-0.02) = 1 + 0.5(-0.02) = 0.99.$$

Example 3.1.12. Without using a pocket calculator, give an approximated value of $\sqrt{177}$.

SOLUTION: Let us consider the function $f(x) = \sqrt{x}$. As base point c, one that would be a perfect square (in order to compute its square root immediately) and that is as close to 177 as it is possible (in order that the increment would be as least as possible). The candidate is $c = 169$, as $\sqrt{169} = 13$. The derivative of f at $c = 169$ is

$$f'(169) = \frac{1}{2\sqrt{169}} = \frac{1}{26}$$

.

So $\sqrt{177} = \sqrt{169+8} = f(169+8) \approx f(169) + f\ '(169).8 = 13 + \frac{8}{26} \approx 13.307$.
The correct value with 3 decimals is 13.304

3.1.10 Implicit differentiation

Definition 3.1.13. An equation $F(x, y) = 0$ defines $y = f(x)$ in an implicit way near the point (x_0, y_0) when it is satisfied that:

1. $F(x_0, y_0) = 0$
2. if (x, y) is close to the point (x_0, y_0) : $F(x, y) = 0 \Longleftrightarrow y = f(x)$.

Theorem 3.1.14. $F(x_0, y_0) = 0$, $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \Longrightarrow$
The equation $F(x, y) = 0$ defines $y = f(x)$ in an implicit way near the point (x_0, y_0) .

Theorem 3.1.15. $y_0' = f\ '(x_0)$ *can be obtained from the equation:*
$\frac{\partial F}{\partial x}(x_0, y_0) + \frac{\partial F}{\partial y}(x_0, y_0)y_0' = 0 \qquad (*)$

This way, even if we do not know the explicit expression of $y = f(x)$, we can have an approximate idea of the function knowing that
$y - y_0 = f\ '(x_0)(x - x_0)$
is the tangent line of $f(x)$ at the point (x_0, y_0).

Besides, if by taking the derivative of the equation $(*)$ we find $y_0" \neq 0$, our information on the function improves since:

1. if $y_0" > 0 \Longrightarrow f$ is convex near $x_0 \Longrightarrow$ the graph of f lies above the tangent line near the point (x_0, y_0).
2. if $y_0" < 0 \Longrightarrow f$ is convex near $x_0 \Longrightarrow$ the graph of f lies below the tangent line near the point (x_0, y_0).

3.2 Some Theorems on Differentiable Functions

3.2.1 Monotonicity

Definition 3.2.1. It is said that the función f is estrictly increasing in an interval I when: for any $x_1 < x_2$ points of I, $f(x_1) < f(x_2)$.

Definition 3.2.2. It is said that the función f is monotone increasing in an interval I when: for any $x_1 < x_2$ points of I, $f(x_1) \leq f(x_2)$.

When we say increasing, we will refer to strictly increasing.
Analogously, we can define

Definition 3.2.3. It is said that the function f is strictly decreasing in an interval I when for any $x_1 < x_2$ points of I, $f(x_1) > f(x_2)$.

Definition 3.2.4. It is said that the function f is monotone decreasing in an interval I when: for any $x_1 < x_2$ puntos de I, $f(x_1) \geq f(x_2)$.

When we say decreasing, we will refer to strictly decreasing.
From these definitions, we can define (analogously for decreasing):

Definition 3.2.5. It is said that the function f is increasing at a point c when when there exists an interval $I = (c - \delta, c + \delta)$ such that, for any $x_1 < x_2$ points of I, $f(x_1) < f(x_2)$.

Definition 3.2.6. It is said that the function f is increasing to the right of c when there exists an interval $I = [c, c + \delta)$ such that, for any $x_1 < x_2$ points of I, $f(x_1) < f(x_2)$.

Definition 3.2.7. It is said that the function f is increasing to the left of c when there exists an interval $I = (c - \delta, c]$ such that, for any $x_1 < x_2$ points of I, $f(x_1) < f(x_2)$

Example 3.2.8.
1. $f(x) = x^3$ is increasing in $\mathbb{R}$ therefore it is increasing at $x = 0$.
2. $f(x) = x^2$ is increasing in $[0, \infty)$, threrefor it is increasing at $x = 0^+$.
3. $f(x) = x^2$ is decreasing in $(-\infty, 0]$, therefore it is decreasing at $x = 0^-$.
4. Obviously, $f(x) = x^2$ is neither increasing nor decreasing at $x = 0$.

Theorem 3.2.9. *If $f'(x)$ is continuous at c and $f'(c) > 0$ ($f'(c) < 0$), then f is increasing (decreasing) at c.*

Note that the theorem is only a sufficient condition, not necessary, because $c = 0$ is a point where $f(x) = x^3$ increases, but $f'(0) = 0$.
Next, we study the increasing/decreasing behavior (i.e., monotonicity) of functions in intervals.

Theorem 3.2.10. *For the differentiable function f to be monotone increasing (decreasing) in an interval I, is necessary and sufficient that for any $x \in I$, $f'(x) \geq 0$ ($f'(x) \leq 0$).*

Theorem 3.2.11. *If $f'(x) > 0$ ($f'(x) < 0$) for any $x \in I$, then f is estrictly increasing (decreasing) in the interval I.*

Example 3.2.12. Find the intervals of monotonicity of $f(x) = 3x - x^3$.

SOLUTION: We have $f'(x) = 3 - 3x^2 = 3(1 - x^2)$. Since $f'(x) > 0$ for $x \in (-1, 1)$ and $f'(x) < 0$ for $x \in (-\infty, 1)$ and $x \in (1, +\infty)$, it follows that f is strictly increasing in $[-1, 1]$ and strictly decreasing in $(-\infty, -1] \cup [1, \infty)$.

3.2.2 Local Extremum of functions

Derivatives are a very useful tool for locating and identifying maximum and minimum values (extremum) of functions. In the following, we suppose that the function f is defined in an open interval $(c-\delta, c+\delta)$ around c.

Definition 3.2.13. The function f has a local maximum (minimum) at the point c if there is $\delta > 0$ such that for every $x \in (c-\delta, c+\delta)$

$$f(x) \leq f(c) \qquad (f(x) \geq f(c)).$$

A local maximum or a local minimum are local extremum of f.

Theorem 3.2.14. *If the function f has an extremum at the point c, then the derivative $f'(c)$ is either zero or does not exist.*

Proof. Without loss of generality, suppose that c a local minimum point of f and that $f'(c)$ exists. By definition of a local minimum, we have $f(c+h) \geq f(c)$ for every h with $|h| < \delta$. Let $h > 0$ and consider the quotient

$$\frac{f(c+h) - f(c)}{h}.$$

It is non–negative and the limit exists when $h \to 0$ and equals $f'(c)$, since f is differentiable at c. Given that the limit of non–negative quantities must be non–negative, we get the inequality $f'(c) \geq 0$. Consider now $h < 0$. Then, the above quotient is non–positive. Taking the limit as $h \to 0$ we get the reverse inequality $f'(c) \leq 0$. Hence it must be $f'(c) = 0$ and we are done. □

The points where the function is not differentiable or the derivative vanishes are possible extrema of f, and for this reason they are called *critical points* of f.

Example 3.2.15. Find the critical points of $f(x) = 3x - x^3$ and $g(x) = |x|$.

Solution: Function f is differentiable at every point and $f'(x) = 3(1-x^2)$. Thus, $f'(x) = 0$ if and only if $x = \pm 1$. Hence, the critical points of f are 1 and -1. Function g is differentiable at every point except $c = 0$, where it has a corner. Actually, the derivative

$$g'(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0, \end{cases}$$

never vanishes. Hence, 0 is the only critical point of g.

Theorem 3.2.16. *Suppose that f is differentiable in an interval $I = (c-\delta, c+\delta)$ around point c (except, maybe, for the point c itself). Then, if the derivative of f changes sign from plus to minus (from minus to plus) when passing through the point c, then f has a local maximum (minimum) at the point c. If the derivative does not change sign when passing through the point c, then the function f does not posses an extremum at the point c.*

Example 3.2.17. Find the the local extrema points of $f(x) = 3x - x^3$ and $g(x) = |x|$.

Solution: We know from Example 3.2.12 that the sign of f' change from minus to plus at -1 and from plus to minus at 1, hence f has a local minimum at -1, and f has a local maximum at 1. On the other hand, g' change from minus to plus at 0 (see Example 3.2.15), hence although g is not differentiable at 0, g has a local minimum at this point.

3.2.3 Theorems of Rolle and Lagrange

Theorem 3.2.18 (Rolle's Theorem)**.** *Let the function f satisfy the following conditions:*

1. *f is continuous on $[a, b]$;*
2. *f is differentiable in (a, b);*
3. *$f(a) = f(b)$.*

Then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

Rolle's Theorem states that there is a point $c \in (a, b)$ such that the tangent line to the graph of the function f at the point $(c, f(c))$ is parallel to the x–axis.

Theorem 3.2.19 (Lagrange's Theorem)**.** *Let the function f satisfy the following conditions:*

1. *f is continuous on $[a, b]$;*
2. *f is differentiable in (a, b).*

Then there is a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Lagrange's Theorem is also known as Mean Value Theorem. It can be interpreted as follows: The number

$$\frac{f(b) - f(a)}{b - a}$$

is the slope of the line r which passes through the points $(a, f(a))$ and $(b, f(b))$ of the graph of f, and $f'(c)$ is the slope of the tangent to the graph of f at $(c, f(c))$. Lagrange's formula shows that this tangent line is parallel to the straight line r.

3.3 L'Hopital's Rule

Now we present a useful technique for evaluating limits that uses the derivatives of the functions involved.

Theorem 3.3.1 (Indetermination of the type 0/0)**.** *Assume that the following conditions are fulfilled:*

1. *The functions f and g are defined and differentiable in an interval $I = (c - \delta, c + \delta)$ around point c (except, maybe, the point c itself);*
2. $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$;
3. *The derivative $g'(x) \neq 0$ for any $x \in I$ (except, maybe, the point c itself).*
4. *There exists the limit $\lim_{x \to c} \dfrac{f'(x)}{g'(x)}$.*

Then,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Example 3.3.2. Evaluate $\lim_{x\to 0}\dfrac{\sin ax}{\tan bx}$, where $a, b\in\mathbb{R}$.

SOLUTION: The limit is of the indeterminate type $0/0$. It is easy to verify that all conditions of Theorem 3.3.1 are fulfilled. Consequently,

$$\lim_{x\to 0}\frac{\sin ax}{\tan bx}=\lim_{x\to 0}\frac{a\cos ax}{\frac{b}{\cos^2 bx}}=\frac{a}{b}.$$

Theorem 3.3.3 (Indetermination of the type $\pm\infty/\infty$)**.** *Assume that (1), (3) and (4) of Theorem 3.3.1 are fulfilled and that (2) is replaced by*

(2') $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=\pm\infty$.

Then,

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Example 3.3.4. Evaluate $\lim_{x\to 0^+} x\ln x$.

SOLUTION: The limit is of the indeterminate type $0\cdot\infty$. Writing $x\ln x$ as $\dfrac{\ln x}{1/x}$ we get an indeterminate form ∞/∞. Applying L'Hospital Rule one obtains

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0.$$

Remark 3.3.5. Similarly to Theorems 3.3.1 and 3.3.3, L'Hopital's Rule can be also stated when $x\to+\infty$ or $x\to-\infty$,

$$\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=\lim_{x\to\pm\infty}\frac{f'(x)}{g'(x)}.$$

Example 3.3.6. Evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$.

SOLUTION: The limit is of the indeterminate type ∞/∞. Differentiating above and below, we get

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0.$$

Remark 3.3.7. Indeterminate forms of other types, $0\cdot\infty$, $\infty-\infty$, 1^∞, 0^0 or ∞^0 can be reduced to indeterminate forms of type $0/0$ or ∞/∞ and to them we can apply L'Hopital's Rule.

Example 3.3.8. Evaluate $\lim_{x\to\infty} x^{1/x}$.

SOLUTION: The limit is of the indeterminate type ∞^0. We represent $x^{1/x}=e^{\ln x/x}$ and study $\lim_{x\to\infty}\dfrac{\ln x}{x}=\lim_{x\to\infty}\dfrac{1}{x}=0$, hence the limit is $e^0=1$.

Example 3.3.9. Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

SOLUTION: The limit is of indeterminate type $\infty-\infty$. If we combine fractions, then we obtain

$$\frac{x\ln x-(x-1)}{(x-1)\ln x},$$

which is of the form $0/0$ at $x=1$. Differentiating above and below, we get

$$\frac{\ln x}{1-x^{-1}+\ln x}$$

which is again $0/0$ at $x=1$. Another differentiation above and below gives

$$\frac{x^{-1}}{x^{-2}+x^{-1}}=\frac{x}{1+x}$$

which has limit $1/2$ as $x\to 1$. Thus, we had to apply L'Hospital Rule twice to find that the limit is $1/2$.

When the hypotheses of the theorems are not satisfied, we might obtain incorrect answers, as in the following example.

Example 3.3.10. Clearly, $\lim_{x\to 0^+}\frac{\ln x}{x}=\frac{-\infty}{0^+}=-\infty$. If we attempt to use L'Hopital Rule, we would obtain

$$\lim_{x\to 0^+}\frac{\ln x}{x}=\lim_{x\to 0^+}\frac{x^{-1}}{1}=+\infty,$$

which is incorrect. Notice that the limit is not indeterminate, hence theorems 3.3.1 and 3.3.3 do not apply.

3.4 Optimization of continuous functions on intervals $[a,b]$

Consider a continuous function f defined on an interval $I=[a,b]$. By Weierstrass' Theorem, f attains in $[a,b]$ global extremum. On the other hand, since a global extremum is also a local extremum, if the global extremum are in the open interval (a,b), then they must be critical points of f. Hence, to locate and classify the global extremum of f, we will use the following recipe:

1. Find the critical points of f in (a,b);
2. Evaluate f at the critical points found in (a) and at the extreme points of the interval, a, b;
3. Select the maximum value (global maximum) and the minimum value (global minimum).

Example 3.4.1. Find and classify the extremum points of $f(x)=3x-x^3$ in the interval $[-2,2]$.

Solution: Since f is continuous and $I = [-2, 2]$ is closed and bounded, by Weierstrass' Theorem f attains on I global maximum and minimum. Then, as explained above, the possible global extremum are among the critical points of f in I and the extreme points of the interval I: -2 and 2. We know that $-1 \in I$ is a local minimizer, $f(-1) = -2$, and $1 \in I$ is a local maximizer, $f(1) = 2$, see Example 3.2.17. On the other hand, $f(-2) = 2$ and $f(2) = -2$, thus points -1 and 2 are both global minimizers of f in I, and points -2 and 1 are global maximizers of f in I.