INTRODUCTION TO TRIGONOMETRY

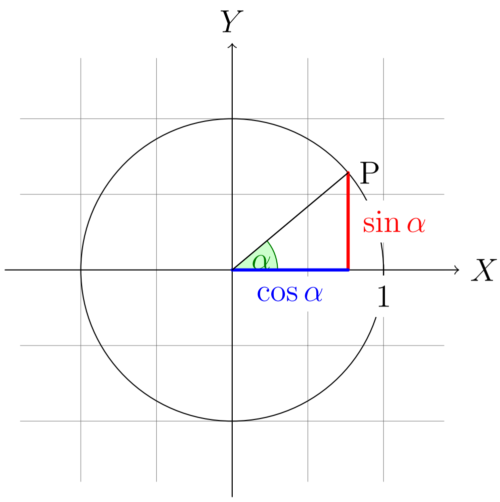
1. Trigonometric functions are those that make a relationship between the wideness of the angles and a quotient of lengths. In short, in the case of angles of amplitude less than 90 degrees, it is the quotient of the lengths of the sides of a rectangle triangle, for the functions that we are going to consider: sine, cosine and tangent.

   Also we are going to study two inverse functions that will be of interest: arc-sine and arc-tangent.

2. How do we measure angles? Generally, we measure a complete twist as 360 degrees (or a quarter of a twist as 90 degrees). But, as we are interested in the easy use of the derivatives of these functions, we will measure angles in radians. What is a radian? It is the angle whose arc has the same length as any of the segments that limit it. If we consider the circumference of radius one, as the length of the circumference will be $2\pi r$, as we take the radius of length one, the circumference will measure 2π or, in other words, 360 degrees are equivalent to 2π radians, 90 degrees are equivalent to $\pi/2$ radians or, analogously, 1 radian is equivalent to a little more than 57 degrees.

3. If we consider an angle of wideness less than 90 degrees, we can consider such angle α limited by the horizontal axis, a segment of positive slope that goes through the origin and the circumference of radius of unit length centered in the origin.

   Then, if we consider the rectangle triangle that has as its hipothenusa, or major side, the segment named before and, as cathetus, or lesser sides, the horizontal and vertical segments originated by the projection of the point P where the hipothenusa cuts the unit circumference on the horizontal axis (see drawing 1).



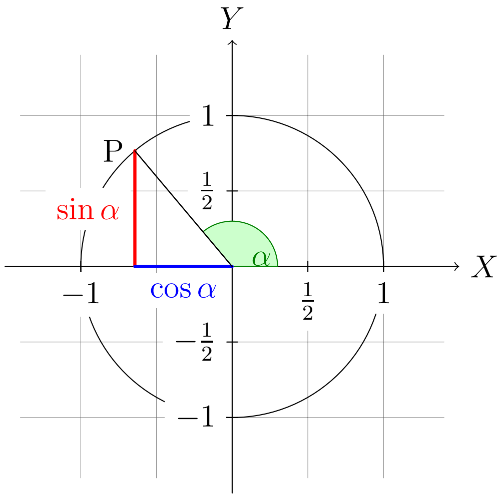
Drawing 1: angle in the first quadrant

   (a) the sine of such angle α is the length of the vertical side.
   (b) the cosine of such angle α is the lengh of the horizontal side.
   (c) the tangent is the quotient between the sine and the cosine or, in other words, the quotient of the length of the vertical side, over the lenght of the horizontal side.

   In this way, we obtain the following vaues (for the measure of angles, the first number is the measure in degrees and the second one measures ei radians).

$\deg s - radians$	$0-0$	$30-\pi/6$	$45-\pi/4$	$60-\pi/3$	$90-\pi/2$
sine	$\sqrt{0}/2=0$	$\sqrt{1}/2=1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2=1$
cosine	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
tangent	0	$1/\sqrt{3}=\sqrt{3}/3$	1	$\sqrt{3}$	not defined

4. If we consider an angle α between 90 and 180 degrees, the best we can do is forgetting the rectangle triangle and think instead of the following lengths, fixed by a segment that goes through the origin and cuts the circumference of radius one in a point P of the second quadrant (see drawing 2):



Drawing 2: angle in the second quadrant

(a) if we descend perpendicularly from the point P until the horizontal axis, the sine of the angle α is equal to the length of such vertical segment. We observe that the sine function takes only positive values or 0.

(b) the cosine of such angle is equal to minus the length determined by the projection of the point P on the horizontal axis and the origin of coordinates. We observe that the cosine function only takes negative values or 0.

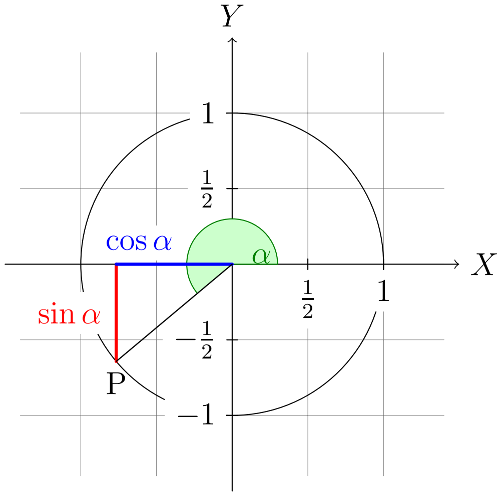
(c) the tangent of such angle α is, once more, the quotient between sine and cosine. We observe that the tangent function only takes negative values or 0.

In this way, the values of the trigonometric functions will be the following ones:

$\deg s - radians$	$90 - \pi/2$	$120 - 2\pi/3$	$135 - 3\pi/4$	$150 - 5\pi/6$	$180 - \pi$
sine	$\sqrt{4}/2 = 1$	$\sqrt{3}/2$	$\sqrt{2}/2$	$\sqrt{1}/2 = 1/2$	$\sqrt{0}/2 = 0$
cosine	0	$-\sqrt{1}/2 = -1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	$-\sqrt{4}/2 = -1$
tangent	not defined	$-\sqrt{3}$	-1	$-\sqrt{3/3}$	0

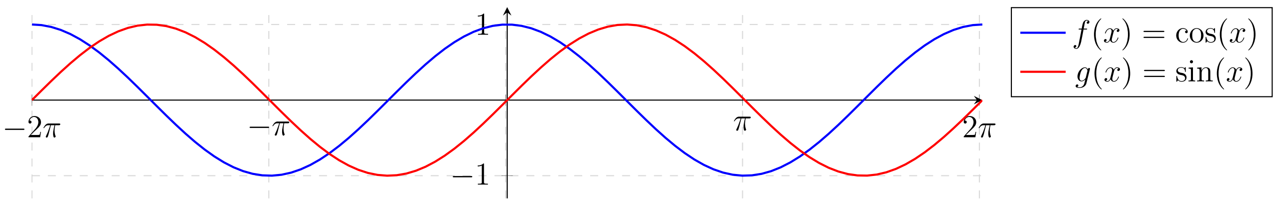
5. If we consider the angle α between 0 and -180 degrees, i.e., twisting the horizontal axis downwards, towards the third and fourth quadrants, the best we can do is to observe that we have results very similar to the ones obtained in sections 3) and 4), observing the following:

sine$(-\alpha)$ = $-$sine(α), cosine$(-\alpha)$ =cosine(α), tangent $(-\alpha)$ = $-$tangent (α) (see drawing 3).



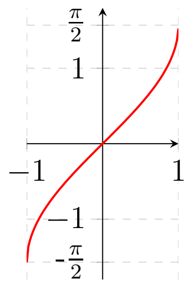
Drawing 3: angle in the third quadrant

6. In order to define these functions for any other angle, it is enough to take into account that both the sine and the cosine are functions of period 2π, (see drawing 4) and that the tangent is a periodic function of period π.



Drawing 4: functions sine and cosine

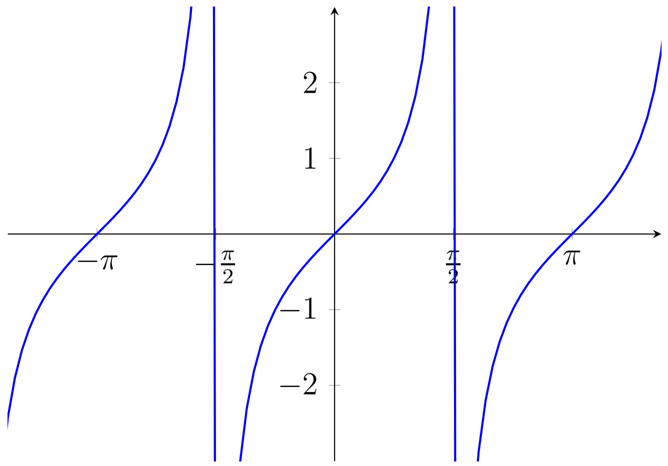
   (a) If we consider the sine function defined from the interval $[-\pi/2, \pi/2]$ until the interval $[-1, 1]$, we observe that such a function is a bijection, so it has an inverse function. See drawing 5.



Drawing 5: function arcsine

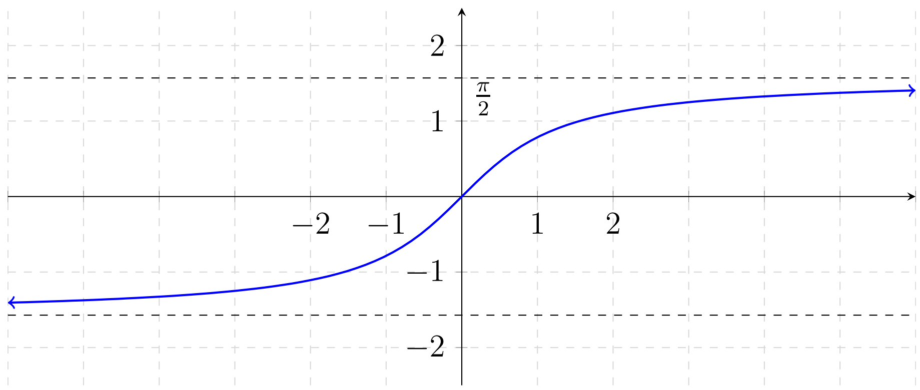
   Such an inverse function, defined from $[-1, 1]$ to $[-\pi/2, \pi/2]$ is called arc-sine.

   (b) If we consider the tangent function defined from the interval $(-\pi/2, \pi/2)$ until the interval $\mathbb{R} = (-\infty, \infty)$, we observe that such a function is a bijection (see drawing 6),



Drawing 6: function tan and its asymptotes

so it has an inverse function. Such an inverse function, definied from $\mathbb{R} = (-\infty, \infty)$ to $(-\pi/2, \pi/2)$, it is called arc-tangent. See drawing 7.



Drawing 7: function arctan and its asymptotes

7. There are many trigonometric identities that are interesting, but we will center just in two:

   (a) $\sin^2(x) + \cos^2(x) = 1$; dividing the previous equality by $\cos^2(x)$, we have:

   (b) $\tan^2(x) + 1 = (1/\cos x)^2$
   The following identities, that make a relationship between sine and cosine of an angle of double length can be useful, although we will rarely use them:

   (c) $\sin(2x) = 2 \sin x \cos x \quad \cos(2x) = \cos^2(x) - \sin^2(x)$