WORKSHEET 4: Applications of the Derivative

1. (*)Calculate the second-order Taylor polynomial at a and find the approximate value of the function using the polynomial at $x = a + 0.1$.

a) $f(x) = e^x$ at $a = 0$ b) $f(x) = \dfrac{\ln x}{x}$ at $a = 1$

a) $P(x) = 1 + x + x^2/2$, so $f(0.1) \approx 1.105$

b)$P(x) = (x-1) - 3\dfrac{(x-1)^2}{2}$, so $f(1.1) \approx 0.085$

2. (*)Given the second-order Taylor polynomial of f at $a = 0$, find out if the function has a local maximum or minimum at the point $(0, f(0))$.

a) $P(x) = 1 + 2x^2$ b) $P(x) = 1 + x + x^2$ c) $P(x) = 1 - 2x^2$

a) f has a local minimum at the point $(0, f(0))$.

b) f has not a local maximum or mínimum at the point $(0, f(0))$.

c) f has a local maximum at the point $(0, f(0))$.

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3. Find the relative and absolute extrema of f in the given intervals:

a)(*) $f(x) = 3x^{2/3} - 2x$ in $[-1, 2]$ b) $f(x) = xe^{-x}$ in $[\frac{1}{2}, \infty)$, $[0, \infty)$ and $\mathbb{R}$

a) i) f obtains a local minimum in x=0 and a local maximum in x=1.

ii) f obtains its absolute minimum in $x = 0$.

iii) f obtains its absolute maximum in $x = -1$.

b) i) f obtains a local and absolute maximum at $x = 1$.

ii) f obtains an absolute minimum, but not a local one, at $x = 0$ on $[0, \infty)$.

iii) f has no local nor absolute minimum when f defined either on $[\frac{1}{2}, \infty)$ or on $\mathbb{R}$.

4. (*)Calculate the point of the graph of $y = -x^3 + 2x^2 + x + 2$ where its tangent line has the greatest slope.

$x = \frac{2}{3}$.

5. (*)The figure A shows the graph of the derivative function of f. Determine the increasing/decreasing and concavity/convexity intervals of f, its local extrema and inflection points.

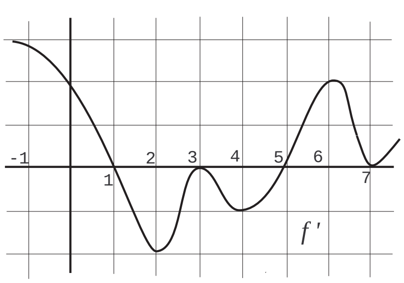

Figure A

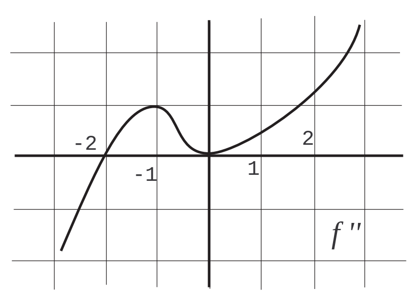

Figure B

f is increasing in $(-\infty, 1]$ and in $[5, \infty)$ and f is decreasing in $[1, 5]$.

So, f obtains a local maximum in 1 and a local minimum in 5. On the other hand,

f is convex in [2,3], [4,6] and in [7,∞) and f is concave in ($-\infty$,2], [3,4] and in [6,7].

So f has inflection points in 2, 3, 4, 6 y 7.

6. The figure B shows the graph of the second derivative function of f. Determine concavity and convexity intervals of f and its inflection points. Determine the monotonicity and local extrema of f assuming that $f'(-3) = f'(0) = 0$.

f is convex on $[-2, \infty)$. f is concave on $(-\infty, -2]$.

Therefore, f has an inflection point in $x = -2$.

Also, f is increasing on $(-\infty, -3]$. And f is decreasing on $[-3, 0]$.

in the same way, f is increasing on $[0, \infty)$.

Therefore, f reaches a local maximum at $x = -3$ and a local minimum at $x = 0$.

7. (*)Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function, and $x > 0$. Check the following inequalities graphically:
$f(1) < \frac{1}{2}\left(f\left(1-x\right) + f\left(1+x\right)\right) < \frac{1}{2}\left(f\left(1-2x\right) + f\left(1+2x\right)\right)$

8. (*)Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a concave function, and $x > 0$. Check the following inequalities graphically:
$f(1) > \frac{1}{2}\left(f\left(1-x\right) + f\left(1+x\right)\right) > \frac{1}{2}\left(f\left(1-2x\right) + f\left(1+2x\right)\right)$

9. Let $f : [0, \infty] \rightarrow \mathbb{R}$ be a convex function such that $f'(1) = 0$

a) Find the local extrema of f.

b) What can be state about the global extrema of f?

c) Suppose now that $f : [0, n] \rightarrow \mathbb{R}$. What can be stated about the global extrema of f?

a) y b): 1 is both local and global minimizer of f

Also, it does not exist a global maximizer of f, since $\lim_{x\to\infty} f(x) = \infty$.

c) In this case besides what we have found regarding the minimizers we know that there will exist a global maximizer at 0 (if $f(n) \leq f(0)$) or at the point n (si $f(0) \leq f(n)$).

10. (*)Given the total cost function $C(x) = 4000 + 10x + 0.02x^2$ and the demand function $p(x) = 100 - \frac{x}{100}$, find the unitary price p that obtains the maximum benefit.

$p = 85$

11. (*)Let $p(x) = x^2 - x + \frac{1}{3}$ be the sale price of one kilo of plutonium when x kilograms are sold. Taking into account that the firm sells a maximum of 2 kilograms on the market, find the value of x that maximizes the profits of the firm. We can assume that the Government pays all costs of the firm.

The maximum income is reached when $x = 2$.

12. (*)Let $p(x) = 100 - \frac{x^2}{2}$ be the demand function of a product and $C(x) = 48 + 4x + 3x^2$ its cost function. What is the production x that minimizes the average cost?

$x = 4$