WORKSHEET 2: Limits and Continuity

1. (*)Calculate:

a) $\lim\limits_{x\to 0} \dfrac{4x^3+2x^2-x}{5x^2+2x}$ b) $\lim\limits_{x\to 2} \dfrac{x^3-x^2-x-2}{x-2}$ c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x}$

d) $\lim\limits_{x\to \infty} \dfrac{x^2-\sqrt{x}}{\sqrt{x^3+3x^4}}$ e) $\lim\limits_{x\to \infty} \dfrac{senx}{x}$ f) $\lim\limits_{x\to -\infty} \dfrac{x^2cosx+1}{x^2+1}$

g) $\lim\limits_{x\to -\infty} \dfrac{3x^3+2x^2+x+2}{x^2-7x+1}$ h) $\lim\limits_{x\to -\infty} \dfrac{x^4-ax^3}{x^2+1}$ i) $\lim\limits_{x\to 0} \dfrac{x^4-x^3}{x^2+b}$

a) $\lim\limits_{x\to 0} \dfrac{4x^3+2x^2-x}{5x^2+2x} = -1/2.$

b) $\lim\limits_{x\to 2} \dfrac{x^3-x^2-x-2}{x-2} = 7$

c) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2}}{x} = \frac{1}{2\sqrt{2}}$

d) $\lim\limits_{x\to \infty} \dfrac{x^2-\sqrt{x}}{\sqrt{x^3+3x^4}} = \dfrac{1}{\sqrt{3}}$

e) $\lim\limits_{x\to \infty} \dfrac{senx}{x} = 0.$

f) $\lim\limits_{x\to -\infty} \dfrac{x^2cosx+1}{x^2+1}$ does not exist.

g) $\lim\limits_{x\to -\infty} \dfrac{3x^3+2x^2+x+2}{x^2-7x+1} = -\infty.$

h) $\lim\limits_{x\to -\infty} \dfrac{x^4-ax^3}{x^2+1} = \infty.$

i) $\lim\limits_{x\to 0} \dfrac{x^4-x^3}{x^2+b} = 0.$

2. Find all discontinuities (if there are any) of the following functions:

a)(*)$f(x) = \dfrac{|x-3|}{x-3}$ b) (*)$g(x) = \begin{cases} \dfrac{2x}{x+1} & if \quad x < -1. \\ e^{1/x} & if \quad -1 \le x < 0 \\ \pi & if \quad x = 0 \\ 1/x & if \quad 0 < x \end{cases}$

a) f is not continuous at $x = 3$.

b) g is not continuous at $x = -1$ nor at $x = 0$.

3. (*)Calculate:

a) $\lim\limits_{x\to -3^+} \dfrac{x^2}{x^2-9}$ b) $\lim\limits_{x\to -3^-} \dfrac{x^2}{x^2-9}$ c) $\lim\limits_{x\to 0^+} \dfrac{2}{senx}$ d) $\lim\limits_{x\to 0^-} (1-1/x)^{\frac{1}{x}}$ e) $\lim\limits_{x\to 0^-} \dfrac{x^2-2x}{x^3}$

a) $\lim\limits_{x\to -3^+} \dfrac{x^2}{x^2-9} = -\infty.$

b) $\lim\limits_{x\to -3^-} \dfrac{x^2}{x^2-9} = \infty.$

c) $\lim\limits_{x\to 0^+} \dfrac{2}{senx} = \infty.$

d) $\lim\limits_{x\to 0^-} (1-1/x)^{\frac{1}{x}} = 0.$

e) $\lim\limits_{x\to 0^-} \dfrac{x^2-2x}{x^3} = -\infty.$

4. Find every asymptote of the following functions:

$$(^*)f(x) = \frac{x^3}{x^2-1} \quad g(x) = \frac{x^2-1}{x} \quad (^*)h(x) = \sqrt{x^2-1} \quad (^*)m(x) = \frac{1}{\ln x} \quad (^*)n(x) = \mathrm{e}^{\frac{1}{x}}$$

a) $y = x$ is oblique asymptote at ∞ (and, analogously, at $-\infty$). On the other hand, $x = -1, x = 1$ are the vertical asymptotes.

b) $y = x$ is oblique asymptote at ∞ (and, analogously, at $-\infty$). On the other hand, $x = 0$ is the only vertical asymptote.

c) $y = x$ is oblique asymptote at ∞. Nevertheless, the asymptote at $-\infty$ is $y = -x$.

d) $y = 0$ is the horizontal asymptote at ∞, $x = 0$ is not a vertical asymptote, $x = 1$ is.

e) $y = 1$ is the horizontal asymptote at ∞ and at -∞, $x = 0$ is the only vertical asymptote.

5. Prove that every odd-degree polynomial has at least one root (zero).

6. (a) (*) Use the Intermediate Value Theorem to prove that the following functions have a zero in the given interval.

i)$f(x) = x^2 - 4x + 3$ in $[2, 4]$; ii)$f(x) = x^3 + 3x - 2$ in $[0, 1]$.

(b) (*) Use the Bisection Method (applying Bolzano's Theorem repeatedly), to obtain the zero with an error of ± 0.25.

x=3 is an exact zero of f. On the other hand, x=3/4 is a zero of g with an error of ± 0.25.

7. (*)Demonstrate that the equations $x^4 - 11x + 7 = 0$ and $2^x - 4x = 0$ have at least two different solutions each.

a) There is a root between 0 and 1, and another root between 1 and 2.

b) There is a root between 0 and 1. On the other hand, $g(4) = 0$.

8. (*)Prove that the equation $x^7 + 3x + 3 = 0$ has only one solution. Round the solution to the nearest whole number.

The function is increasing. For that reason the solution, it it exists, it will be unique.

On the other hand, the integer part of the root is -1, because $f(-1) < 0 < f(0)$.

9. Let f and g be continuous functions on $[a, b]$ and $f(a) < g(a)$, $f(b) > g(b)$, prove that $x_0 \in (a, b)$ exists such that $f(x_0) = g(x_0)$

10. (*) Discuss in the following cases if the functions attain their global and/or local extrema in the given interval:

a) $f(x) = x^2 \quad x \in [-1, 1]$ b) $f(x) = x^3 \quad x \in [-1, 1]$

a) f obtains a global maximum on -1 and 1. It has no local maxima.

f obtains a local and global minimum on 0.

b) g obtains a global minimum on -1 an a global maximum on 1.

It has no local extrema.

11. Change the given interval in the previous exercise by $[0, \infty)$ firstly and then by $\mathbb{R}$.

On $[0, \infty)$: neither f nor g obtain global maxima nor local extrema. f and g obtain global minimum on 0.

On $\mathbb{R}$: f obtains local and global minimum on 0. g does not obtain local or global minimum. Neither f nor g obtain global or local maximum.