Chapter 4

Applications of the derivatives

4.1 Higher order derivatives

If the derivative f' of the function f is defined in an interval $(c-\delta, c+\delta)$ around point c, then the *second derivative* of f is the derivative of the function f', and it is denoted f''. The *third derivative* is defined as the derivative of the second derivative and so on. The third derivative is denoted f''' and, more generally, the nth order derivative by $f^{(n)}$, and once the $(n-1)$th derivative is computed, it is given by $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$.

We will say that a function is of class C^n if the nth order derivative of f, $f^{(n)}$, exists in an open interval, and $f^{(n)}$ is continuous.

Example 4.1.1. Given the function $f(x) = 4x^4 - 2x^2 + 1$, $f'(x) = 16x^3 - 4x$, $f''(x) = 48x^2 - 4$, $f'''(x) = 96x$, $f^{(4)}(x) = 96$ and $f^{(n)}(x) = 0$ for every $n \geq 5$.

4.2 Taylor polynomial

4.2.1 Taylor polynomial of order 2

Remark: the tangent line or Taylor polynomial of order 1:

$$y = P_{1,a}(x) = f(a) + f'(a).(x-a)$$

is characterized by the fact that satisfies :

$$\lim_{x\to a} \frac{f(x) - P_{1,a}(x)}{(x-a)} = 0$$

what can be proven using L'Hopital rule.

From the limit above we can define the Taylor polynomial.

Definition 4.2.1. The Taylor polynomial of order n is characterized as the unique polynomial of degree $\leq n$ that satisfies: $\lim\limits_{x\to a} \frac{f(x)-P_{n,a}(x)}{(x-a)^n} = 0$.

From the limit above can be deduced that, when $n = 2$, :

Theorem 4.2.2. $P_{2,a}(x) = f(a) + f'(a).(x-a) + \frac{f"(a)}{2}(x-a)^2$

Proof : Use L'Hopital rule.

Remark: the first and second derivatives of the Taylor polynomial of order 2 at point $x = a$ coincide with those of f.

4.2.2 Second order approximation

The Taylor polynomial is the tangent parabola to f (if $f"(a) \neq 0$). What is the Taylor polynomial good for if $f"(a) \neq 0$? In other words, what is the tangent parabola used for?

1. To know the relative position of the graph of f with respect to the tangent line.
2. Also, if $f\,'(a) = 0$, to study local extrema by the sign of $f"(a)$.

   Let us assume that $f'(a) = 0, f"(a) \neq 0$. If the polynomial has a local extremum, f does as well. Obviously, if the function does not have it, neither does the polynomial.

   See also section 4.3.
3. To obtain better approximations.

   Example 4.2.3. Find an approximated value of $\ln(0,9)$ and $\ln(1,2)$ using:

   a) the Taylor polynomial of $f(x) = \ln(1 + x)$ at $a = 0$: $\ln(1 + x) \approx x - x^2/2$; or

   b) the Taylor polynomial of $f(x) = \ln(x)$ at $a = 1$: $\ln(x) \approx (x - 1) - (x - 1)^2/2$

4.3 Second order optimality conditions

Let f be a function of class C^2.

Necessary conditions

- $f(c)$ is a local minimum of $f \Rightarrow f''(c) \geq 0$;
- $f(c)$ is a local maximum of $f \Rightarrow f''(c) \leq 0$.

Sufficient conditions

Let c be a critical point, $f'(c) = 0$.

- $f''(c) > 0 \Rightarrow f(c)$ is a (strict) local minimum of f;
- $f''(c) < 0 \Rightarrow f(c)$ is a (strict) local maximum of f.

Example 4.3.1. Let $f(x) = 4x^4 - \frac{8}{3}x^3 + 1$. we study local extrema with the first and second derivative. We have that $f'(x) = 16x^3 - 8x^2$ and $f''(x) = 48x^2 - 16x$. Critical points are $x = 0$ and $x = \frac{1}{2}$. Since $f''(0) = 0$, we cannot conclude anything by the second derivative test. We have that, $f''(\frac{1}{2}) = \frac{48}{4} - \frac{16}{2} = 12 - 8 = 4 > 0$, therefore $\frac{1}{2}$ is a local minimizer of f. In order to tell what type of point 0 is, we can resort to the first derivative test, since $f'(-1) < 0$, $f'(1/4) < 0$, it follows that f decreases when $x < \frac{1}{2}$ and, therefore, $x = 0$ it is neither a local maximizer nor a local minimizer.

Example 4.3.2. Let $f(x) = 4x^4 - 2x^2 + 1$, so $f'(x) = 16x^3 - 4x$ and $f''(x) = 48x^2 - 4$. can point $c = 0$ be a local minimizer of f? No, since $f''(0) = -4 < 0$. Is $c = 0$ a local maximizer of f? Yes, since it is a critical point, $f'(0) = 0$ and $f''(0)$ is negative as we have computed above. Does f have other extremal points? Let us find all its critical points: $f'(x) = 0$ if and only if $x = 0$, $x = \pm\frac{1}{2}$. Now, $f''(\pm\frac{1}{2}) = 8 > 0$, thus both $\frac{1}{2}$ and $-\frac{1}{2}$ are local minimizers.

4.4 Convexity and points of inflection of a function

Assume that the function f has a finite derivative at every point of the interval (a, b). Then, at every point in (a, b) the graph of the function has a tangent which is nonparallel to the y–axis.

Definition 4.4.1. The function f is said to be convex (concave) in the interval (a, b) if, within (a, b), the graph of f lies not lower (not higher) than any tangent.

Theorem 4.4.2 (Characterization of the convexity or concavity by the derivative)**.**

1. *f is convex on the interval I if and only if its derivative increases on I .*
2. *f is concave on the interval I if and only if its derivative decreases on I*

Theorem 4.4.3 (A sufficient condition for convexity/concavity)**.** *If f has second derivative in the interval (a, b) and $f''(x) \geq 0$ ($f''(x) \leq 0$) for every $x \in (a, b)$, then f is convex (concave) in (a, b).*

Theorem 4.4.4 (Global Extrema of concave/convex functions)**.**

1. *If f is convex on I and c is a critical point of f, then c is a global minimizer of f on I.*
2. *If f is concave on I and c is a critical point of f, then c is a global maximizer of f on I.*

Definition 4.4.5. A point c is a point of inflection of the function f if at this point the function changes the curvature, from convex to concave or from concave to convex.

Theorem 4.4.6 (A necessary condition for inflection point)**.** *If f has an inflection point at c and f'' is continuous in an interval around c, then $f''(c) = 0$.*

Theorem 4.4.7 (A sufficient condition for inflection point)**.** *If f'' exists in an interval around c, with $f''(c) = 0$, and the signs of f'' are different on the left and on the right of the point c, then c is an inflection point of f.*

Example 4.4.8. Find the intervals of concavity/convexity of $f(x) = (x + 6)^3(x - 2)$, and the possible inflection points.

SOLUTION: The domain of f is the whole real line, and the function is continuous.

$$f'(x) = 3(x + 6)^2(x - 2) + (x + 6)^3 = (x + 6)^2(3(x - 2) + (x + 6)) = (x + 6)^2(4x),$$
$$f''(x) = 8(x + 6)x + 4(x + 6)^2 = 4(x + 6)(2x + (x + 6)) = 12(x + 6)(x + 2).$$

Hence, $f'' \geq 0$ in the region $x \geq -2$ and in the region $x \leq -6$, and $f'' \leq 0$ in the complement set, $[-6, -2]$. We conclude that f is convex in the interval $(-\infty, -6]$ and in the interval $[-2, +\infty)$, and it is concave in the interval $[-6, -2]$. Obviously, -6 and -2 are inflection points.

4.5 Applications of the derivative to revenue, cost and profit functions of a firm

4.5.1 Revenue, cost and marginal profit

In applied economics, the marginal cost of production is the change in total production cost that comes from making or producing the last unit. Therefore, if the level of production of a company is x, the marginal cost of production is calculated by using the following formula:

$$C(x) - C(x-1) = C'(\alpha_x), \text{ where } \alpha_x \in (x-1, x), \text{ according to Lagrange's Theorem.}$$

On the other hand, the marginal cost of production is also considered the change in total production cost that comes from making or producing one additional unit. Therefore, if the level of production of a company is x, the marginal cost of production is calculated by using the following formula:

$$C(x+1) - C(x) = C'(\beta_x), \text{ where } \beta_x \in (x, x+1), \text{ according to Lagrange's Theorem.}$$

In this subject, when we talk about marginal cost we are always going to consider it as the derivative of the cost function. If we agree that the derivative of the cost function is quite stable, the three different concepts of the marginal cost have a very approximate value. So, we can assume:

$$C'(\alpha_x) \approx C'(x) \approx C'(\beta_x).$$

Notice: The approximation above can be refined as:

$$C(x) - C(x-1) = C'(\alpha_x) < C'(x) < C'(\beta_x) = C(x+1) - C(x).$$

assuming that $C(x)$ is convex, which is quite common. Therefore, its derivative is increasing.

We can do the same as the cost function with the profit function:

$$B(x) - B(x-1) = B'(\alpha_x) \approx B'(x) \approx B'(\beta_x) = B(x+1) - B(x).$$

Analogously, those approximations can be refined as:

$$B(x) - B(x-1) = B'(\alpha_x) > B'(x) > B'(\beta_x) = B(x+1) - B(x)$$

if we assume that $B(x)$ is a concave function, which is quite common. Therefore, its derivative is decreasing.

4.5.2 Company behaviour: minimizing the average cost-maximizing the profit

a) $\dfrac{C(x)}{x}$, the average cost function is convex in general, then its minimizer will be attained at the point:

   x_0 so that: $\left(\frac{C(x)}{x}\right)'(x_0) = 0$.

   Notice: x_0 must be positive, greater than the minimum production (whenever it exists) and less than the maximum production (if it exists).

   Notice: a company that aims at minimizing the average cost, is really looking for the maximization of the probability of having profits, since it has uncertainty about the selling price of its products. Bearing in mind that the company has profits when

   $B(x) = x \cdot p(x) - C(x) > 0 \Longleftrightarrow p(x) > \frac{C(x)}{x}$,

   this means that the company looks for the value of $\frac{C(x)}{x}$ to be as little as possible.

b) $B(x)$, the profit function, is generally concave so its maximum is attained at the point x_0 that verifies: $B'(x_0) = 0$.

   Notice: x_0 must be positive, greater than the minimum production (whenever it exists) and less than the maximum production (if it exists).

   Notice: a company that aims at maximizing its profit has a true knowledge of the selling price $p(x)$ for the production x.

   Only companies with a close monopolistic position in the market can really choose that price.

Recall the concepts of revenue function R, cost function C, and profit function Π of a firm given in the lesson about continuity of functions. Also remember that $P(x)$ represents the market inverse demand function, and x is the quantity of the commodity produced and sold by the firm. We consider three different optimization problems.

<u>Owner's Problem</u>: to maximize profits

$$\max \Pi(x) \quad \text{subject to } x \text{ being feasible.}$$

<u>Sales Manager Problem</u>: to maximize revenue

$$\max R(x) \quad \text{subject to } x \text{ being feasible.}$$

<u>Production Manager Problem</u>: to minimize average cost

$$\min \frac{C(x)}{x} \quad \text{subject to } x > 0 \text{ being feasible.}$$

Let

$$P(x) = A - Bx;$$
$$C(x) = c + ax + bx^2,$$

where A, B, b, c are non-negative, with $A > 0$, $B > 0$, $b > 0$ and $A \geq a$. We have

$$R(x) = xP(x) = x(A - Bx);$$
$$\Pi(x) = R(x) - C(x) = x(A - Bx) - (c + ax + bx^2);$$
$$\overline{C}(x) = \frac{C(x)}{x} = \frac{c}{x} + a + bx.$$

Suppose that there is no production constraints, so that the good can be produced in any quantity.

- Owner's Problem.

$$\Pi'(x) = A - 2Bx - a - 2bx = 0 \Rightarrow x^* = \frac{A - a}{2(B + b)}.$$

  Since

$$\Pi''(x) = -2(B + b) < 0,$$

  the profit function is strictly concave, thus x^* maximizes profits (unique global maximum).

- Sales Manager Problem.

$$R'(x) = A - 2Bx = 0 \Rightarrow x^{**} = \frac{A}{2B}.$$

  Since

$$R''(x) = -2B < 0,$$

  the revenue function is strictly concave, thus x^{**} maximizes revenue (unique global maximum).

- Production Manager Problem.

$$\overline{C}'(x) = -\frac{c}{x^2} + b = 0 \Rightarrow x^{***} = \sqrt{\frac{c}{b}}.$$

  Since

$$\overline{C}''(x) = \frac{2c}{x^3} > 0,$$

  the average cost function is strictly convex, thus x^{***} minimizes average cost (unique global minimum).