# Chapter 1

# The real line. Functions of one variable.

We work in the field of real numbers, $\mathbb{R}$. It is a totally ordered set of numbers defined by the inequalities.

## 1.1 Inequalities in the real line

Geometrically, $x < y$ means $x$ is to the left of $y$.

Properties:

1. $a < b \Longleftrightarrow a + K < b + K$,
2. If $K > 0:\ a < b \Longleftrightarrow aK < bK$. If $K < 0:\ a < b \Longleftrightarrow aK > bK$
3. $a < b, c < d \Rightarrow a + c < b + d;\ 0 \leq a < b, 0 \leq c < d \Rightarrow ac < bd$
4. $ab > 0 \Longleftrightarrow [a > 0, b > 0 \text{ o } a < 0, b < 0];\ ab < 0 \Longleftrightarrow [a > 0, b < 0 \text{ o } a < 0, b < 0]$
5. if $n$ es an even number: $x^n > 0 \Longleftrightarrow x \neq 0$; if $n$ is an odd number: $x^n > 0 \Longleftrightarrow x > 0$.

**Definition 1.1.1.** $I \subset \mathbb{R}$ is an Interval $\Longleftrightarrow$ $[\forall x, y \in I, \forall z \in \mathbb{R},\ x < z < y \Rightarrow z \in I]$ and Cardinal$(I) > 1$.

That is to say, an interval contains a least two points and all points in between them.

Intervals can be closed (if they contain all boundary points) or open (if they do not contain any of the boundary points).

Intervals can be bounded (if they do not approach $\pm\infty$) or unbounded (if they approach $\pm\infty$).

## 1.2 Absolute value

For the study of the properties of functions we need the concept of absolute value of a number.

**Definition 1.2.1.** Let $a$ be a real number. Then the absolute value of $a$ is the real number $|a|$ defined by

$$|a| = \begin{cases} -a, & \text{if } a < 0 \\ a, & \text{if } a \geq 0 \end{cases}.$$

The absolute value $|a|$ can be interpreted as its distance from zero on the number line. An alternative definition is $|a| = \sqrt{a^2}$.

### 1.2.1 Properties

Let $a, b \in \mathbb{R}$.

1. $|a| \geq 0$, and $|a| = 0$ if and only if $a = 0$.
2. $-|a| \leq a \leq |a|$.
3. $|ab| = |a||b|$.
4. If $a^2 \leq b^2$, then $|a| \leq |b|$.
5. (The Triangle Inequality)
$$|a + b| \leq |a| + |b|.$$
6. Let $p$ any positive number. Then
   (a) $|a| \leq p$ if and only if $-p \leq a \leq p$.
   (b) $|a| \geq p$ if and only if $a \geq p$ or $a \leq -p$.

**Example 1.2.2.** Prove the Triangle Inequality.

Solution:

$$|a+b|^2 = (a+b)^2 = a^2 + 2ab + b^2 = |a|^2 + 2ab + |b|^2 \leq |a|^2 + 2|a|\,|b| + |b|^2 = (|a| + |b|)^2,$$

from which we conclude $|a + b| \leq |a| + |b|$.

**Example 1.2.3.** Solve the inequality $5 < |2x - 1| \leq 9$.

Solution: From (6b), $5 < |2x - 1|$ if and only if $2x - 1 > 5$ or $2x - 1 < -5$; hence, $x > 3$ or $x < -2$, that gives the set $(-\infty, -2) \cup (3, +\infty)$. From (6a), $|2x - 1| \leq 9$ if and only if $-9 \leq 2x - 1 \leq 9$; hence $-4 \leq x \leq 5$, that is, $x \in [-4, 5]$. Both inequalities hold simultaneously if and only if $x$ belongs to $(-\infty, -2) \cup (3, +\infty)$ and to $[-4, 5]$. Hence, the solution set is $[-4, -2) \cup (3, 5]$.

## 1.3 Functions

A *function* $f$ consists of two sets, $D$ and $R$, called the *domain* and the *range* of $f$, and a rule that assigns to each element $x$ of $D$ exactly one element $y$ of $R$, called the image of $x$ , or value of the function at $x$. This is expressed as $f : D \longrightarrow R$.

The *image set* of the function $f$ is the set $\mathrm{Im}(f) = \{f(x) \,|\, x \in D\}$. Note that the image set of $f$ is a subset of the range, $\mathrm{Im}(f) \subseteq R$.

The *graph* of $f$ is the set $G(f)$ of ordered pairs $(x, y)$ such that $x$ is in the domain of the function and $y$ is the corresponding element in the range. The image of the function at $x \in D$ is the element $y \in R$ such that $(x, y) \in G$ and will be denoted $y = f(x)$. Thus,

$$G(f) = \{(x, f(x)) \,|\, x \in D\}.$$

For instance, if $D = \{1, b, e\}$ and $R = \{-1, h, k, 0\}$, the rule $f$ that assigns $1 \mapsto h$, $b \mapsto h$, $e \mapsto 0$ is a function, but the rule $g$ assigning $1 \mapsto h$, $b \mapsto h$, $1 \mapsto -1$ is not. Note in the first example that not every element of the range is assigned an element of the domain; in fact $\mathrm{Im}(f) = \{h, 0\}$, that is strictly smaller than the range. Also, $G(f) = \{(1, h), (b, h), (e, 0)\}$.

Most often, we will write the domain $D$ of $f$ as $D(f)$.

**Definition 1.3.1.** Let $f : D \longrightarrow R$ be a function. We say that $f$ is *one-to-one* or injective if different elements of $D$ have different images by $f$, that is

$$x_1, x_2 \in D, \quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

We say that $f$ is *onto* or suprajective if every element of $R$ is the image by $f$ of some element of $D$, that is, $\mathrm{Im}(f) = R$, or in other words, for every $y$ in $R$, there is $x \in D$ such that $y = f(x)$.

The function of the example above, with $f(1) = f(b) = h$ and $f(e) = 0$ is neither one-to-one nor onto.

We are specially interested in real functions of one variable with $D, R \subset \mathbb{R}$.

**Example 1.3.2.** It costs $1 + 1/x$ euros per liter to manufacture $x$ liters of a detergent. What is the total cost of manufacturing 100 liters? What is the profit in manufacturing $x$ liters of detergent if it is sold at 2 euros per liter?

SOLUTION: The cost of manufacturing 100 liters is $100 \times (1 + 1/100) = 101$ euros. Let $C(x)$ be the cost of manufacturing $x$ liters; then $C(x) = x(1 + 1/x) = x + 1$. The profit $\Pi(x)$ is equal to sales minus cost thus, $\Pi(x) = 2x - C(x) = x - 1$. In particular, the profit of manufacturing 100 liters is $\Pi(100) = 99$ euros.

In this example, the graph of both the cost and the profit function are straight lines.

**Example 1.3.3.** Citizens of country A pay taxes according to their yearly income, $w$ (in some given monetary units, m.u.). For income $w \leq 10$, the individual pays nothing; for income $10 < w \leq 30$, the tax rate is $\tau = 18\%$ applied on the difference $w - 10$, while an income $w > 30$ pays a tax rate of $t = 18\%$ for the first 20 m.u. in excess of 10 m.u., and a tax rate of $t' = 24\%$ for the rest of the amount. Write the tax function. What is the total tax paid by someone who earns $w = 40$ in a year? What is the effective tax rate paid by this person (taking into account that the first 10 m.u. are not taxed)?

Solution: The tax function is piecewise defined.

$$T(w) = \begin{cases} 0, & \text{if } w \leq 10; \\ 0.18(w-10), & \text{if } 10 < w \leq 30; \\ 0.18 \times 20 + 0.24(w-30), & \text{if } w > 30; \end{cases}$$

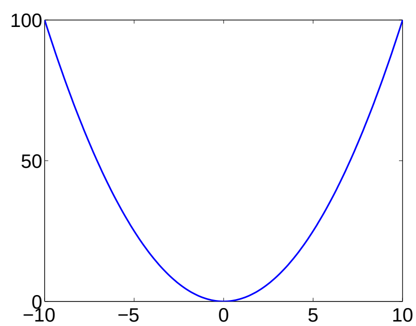
$T(40) = 0.18 \times 20 + 0.24(40-30) = 3.6 + 2.4 = 6$ m.u. The effective tax rate for a person with income $w > 10$ is

$$\tau(w) = \frac{T(w)}{w-10} \times 100\%.$$

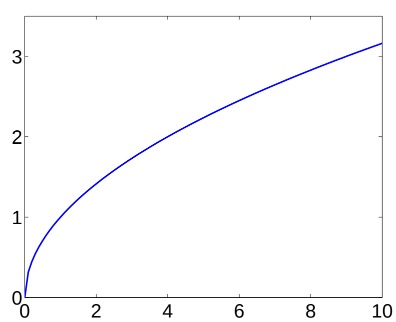
Hence, in this case, $\tau(40) = \frac{6}{30} \times 100 = 20\%$.

**Example 1.3.4.**

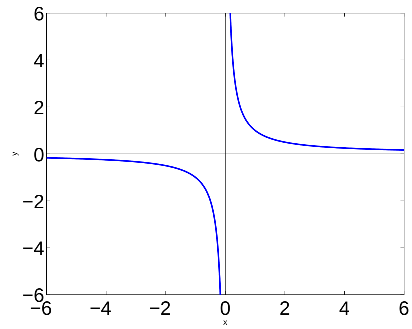
1. Let $f(x) = x^2$. The domain is $D = \mathbb{R}$ and the range is $R = [0, +\infty)$. The graph is a parabola passing through $(0,0)$ that opens upward.



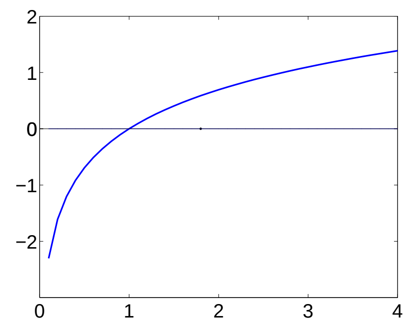

2. Let $g(x) = \sqrt{x}$. The domain is $D(g) = [0, +\infty)$ and the range is $R(g) = [0, +\infty)$. The graph is shown below.



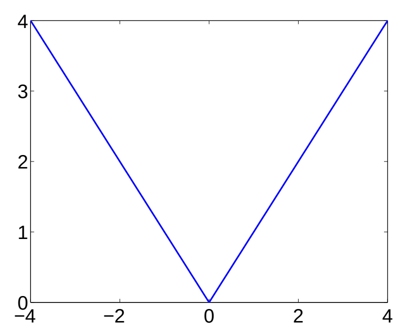

3. Let $h(x) = 1/x$. The domain is $D(h) = \mathbb{R} - \{0\}$ and the range is $R(h) = \mathbb{R} - \{0\}$. The graph is shown below.



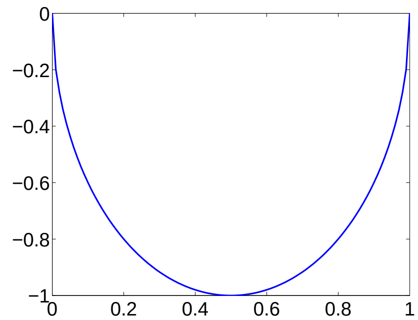

4. Let $l(x) = \ln x$. The domain is $D(l) = (0, +\infty)$ and the range is $R(l) = \mathbb{R}$. The graph is shown below.




5. Let $m(x) = |x|$. The domain is $D(m) = \mathbb{R}$ and the range is $R(m) = [0, +\infty)$. The graph is shown below.




6. Let $n(x) = -2\sqrt{x(1-x)}$. The domain is $D(n) = [0, 1]$ and the range is $R(n) = [-1, 0]$. The graph is shown below.

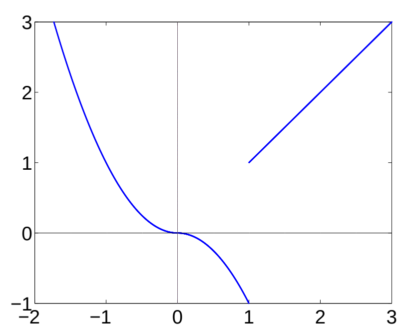



The domain of a function is clear from the statement of the function. The range, however, is not always easily determined. The graph of the function helps to visualize the range. If you project lights horizontally toward the $y$–axis from a great distance away, the shadow of the graph on the $y$–axis gives a picture of the range.

A function can be defined by different formulas for different intervals. As an example, the graph of the function

$$f(x) = \begin{cases} x^2, & \text{if } x \le 0; \\ -x^2, & \text{if } 0 < x \le 1; \\ x, & \text{if } x > 1. \end{cases}$$

is shown below.




### 1.3.1 Revenue, cost, and profit functions

Consider a firm that produces a good in quantity $x \ge 0$ and sells the good in a market where the inverse demand function, or unitary price of the good is $P(x)$. The *revenue* function of the firm is

$$R(x) = xP(x).$$

The production of $x$ units of the good costs to the firm $C(x)$ monetary units; this is the *cost* function of the firm. The firm's *profit* function is the difference between revenue and cost

$$\Pi(x) = R(x) - C(x) = xP(x) - C(x).$$

The firm's *average cost* is

$$\overline{C}(x) = \frac{C(x)}{x},$$

that is the average cost of each of the $x$ units produced.

Typical examples are: $P(x) = 100 - 2x$ if $x \leq 50$, $P(x) = 0$, otherwise; $C(x) = 200 + \frac{1}{2}x^2$ (here, the firms faces a fixed cost of 200 m.u., independently of the production level).

Hence

$$R(x) = xP(x) = x(10 - 2x),\ (x \leq 50); \quad R(x) = 0,\ (x > 50),$$

is the revenue function.

$$\Pi(x) = R(x) - C(x) = x(100 - 2x) - 200 - \frac{x^2}{2},\ (x \leq 50); \quad \Pi(x) = -200 - \frac{x^2}{2},\ (x > 50),$$

is the profit function.

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{200}{x} + \frac{x}{2},\ (x > 0),$$

is the average cost. We can answer the following questions:

- How much is the average cost of producing 20 units of good?

  It is $\overline{C}(20) = \frac{200}{20} + \frac{1}{2}20 = 20$.

- Which is the revenue?

  It is $R(20) = 20(100 - 2 \cdot 20) = 1200$ m.u.

- Which is the profit?

  It is $\Pi(20) = R(20) - C(20) = 1200 - (200 + \frac{20^2}{2}) = 1200 - 400 = 800$ m.u.

We still cannot answer the following interesting questions:

(i) Which is the optimal production level maximizing profits? This would be considered as the objective of the owners of the firm.

(ii) Which is the optimal production level maximizing revenue? This could be considered as the objective of the sales manager.

(iii) Which is the optimal production level minimizing average cost? This could be considered as the objective of the production manager.

We will learn along the course how to solve these, and other related problems.

### 1.3.2 Operations with functions

Consider two functions $f, g : D \subset \mathbb{R} \longrightarrow R$ and a $\lambda \in \mathbb{R}$.

1. The sum of $f$ and $g$ is the function defined as $(f+g)(x) = f(x) + g(x)$.
2. The product of function $f$ by a scalar $\lambda$ is defined as the function $(\lambda f)(x) = \lambda f(x)$.
3. The product of $f$ and $g$ is the function defined as $(fg)(x) = f(x)g(x)$.
4. The quotient of $f$ and $g$ is the function defined as $(f/g)(x) = f(x)/g(x)$ whenever $g(x) \neq 0$.

### 1.3.3 Composition of functions

Let functions $f : D(f) \longrightarrow \mathbb{R}$ and $g : D(g) \longrightarrow \mathbb{R}$ such that $R(f) \cap D(g) \neq \emptyset$. Observe that we have imposed that $g$ is well defined at some points of the range of $f$.

**Definition 1.3.5.** The composition of functions $f$ and $g$ is the function $g \circ f : D \longrightarrow \mathbb{R}$ defined as

$$(g \circ f)(x) = g(f(x)).$$

**Example 1.3.6.** If $f(x) = \sqrt{4 - x^2}$ and $g(x) = 2x$, find $g \circ f$, $f \circ g$ and their domains and ranges.

Solution: By definition

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(\sqrt{4-x^2}) = 2\sqrt{4-x^2},\\ (f \circ g)(x) &= f(g(x)) = f(2x) = \sqrt{4-(2x)^2} = 2\sqrt{1-x^2}.\end{aligned}$$

Observe that $g \circ f \neq f \circ g$. Finally

$$\begin{aligned}D(g \circ f) &= [-2, 2],\\ D(f \circ g) &= [-1, 1].\end{aligned}$$

and (why?)

$$\begin{aligned}R(g \circ f) &= [0, 4],\\ R(f \circ g) &= [0, 2].\end{aligned}$$

### 1.3.4 Inverse function

The function $I(x) = x$ is called the *identity* function. We will define the inverse of a function $f$ as the function $g$ (if any) such that $f \circ g = g \circ f = I$.

**Definition 1.3.7.** The inverse of the function $f : D \longrightarrow \mathbb{R}$ is denoted $f^{-1}$ and satisfies $f^{-1}(f(x)) = x \ \forall x \in Dom(f)$ and $f(f^{-1}(x)) = x \ \ \forall x \in \mathrm{Im}(f) = R(f)$

Observe that the inverse has the following properties:

1. $y = f(x)$ if and only if $x = f^{-1}(y)$;

2. It is unique;

3. The domain of $f^{-1}$ is the range of $f$;

4. The range of $f^{-1}$ is the domain of $f$.

**Example 1.3.8.** If $f(x) = \frac{x-1}{2}$ and $g(x) = 2x+1$, $f$ and $g$ are inverse functions. Actually,

$$g(f(x)) = g\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1 = x$$
$$f(g(x)) = f(2x+1) = \frac{(2x+1)-1}{2} = x.$$

**Example 1.3.9.** Find the inverse of $f(x) = 4 - x^2$ for $x \geq 0$.

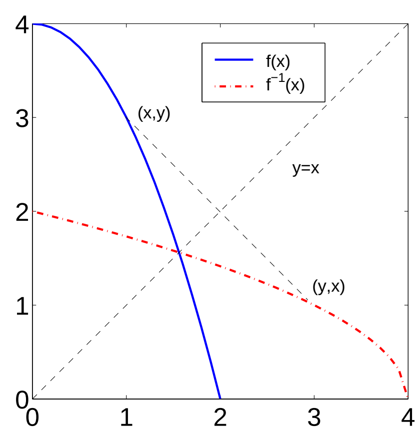
SOLUTION: The inverse of $f$ can be determined as follows:

1. Set $y = 4 - x^2$.

2. Solve for $x$, $x = \sqrt{4-y}$ (we disregard the negative root since $x \geq 0$).

Then $f^{-1}(y) = \sqrt{4-y}$, with domain $(-\infty, 4]$. We can rename variables, and to write $f^{-1}$ as a function of the "usual" variable $x$, $f^{-1}(x) = \sqrt{4-x}$.

The graphs of $f$ and its inverse are shown in the figure below.




Observe that for every point $(x, y)$ in the graph of $y = f(x)$, there is a point $(y, x)$ in the graph of $y = f^{-1}(x)$. This occurs because whenever $f(x) = y$, then $f^{-1}(y) = x$ by the definition of inverse. Hence, if we fold the coordinate plane along the line $y = x$, then the graphs of $f$ and $f^{-1}$ will coincide. The line $y = x$ is the line of symmetry.

**Remark 1.3.10.** Not all functions have inverse functions. Functions that do have inverses are called *one–to–one functions*, meaning that to each $y$–value corresponds exactly one $x$–value. One–to–one functions can be identified by the *horizontal–line test:* If the graph of $f$ is such that no horizontal line intersects the graph in more than one point, then $f$ is one–to–one and admits inverse.

**Example 1.3.11.** The function $y = f(x) = 4 - x^2$ is not one–to–one in the whole real line, since to $y = 3$ corresponds two different values of $x$, namely $x = 1$ and $x = -1$: $f(1) = f(-1) = 3$. Hence, there is no inverse of $f$ in $\mathbb{R}$. This is the reason we considered the function in the region $x \geq 0$ in the previous example. The function fails to pass the horizontal–line test in $\mathbb{R}$ but the test is positive in the region $x \geq 0$ and $f$ admits an inverse in $[0, \infty)$.

### 1.3.5 Periodic functions and symmetry

A function $f$ is *periodic* with *period* $p > 0$ if for every $x$ in the domain of $f$, $f(x+p) = f(x)$ (actually, the period is the smallest $p > 0$ with this property). As we know, $\sin(x)$ and $\cos(x)$ are periodic functions of period $2\pi$, and $\tan(x)$ is of period $\pi$.

In general, if a function $f(x)$ is periodic with period $p$, then $g(x) = f(cx)$ is periodic with period $\frac{p}{|c|}$ $(c \neq 0)$. This is easily shown as follows:

$$g(x + \frac{p}{|c|}) = f(c(x + \frac{p}{|c|})) = f(cx \pm p) = f(cx) = g(x).$$

Thus, the function $\sin 2x$ is periodic with period $\pi$.

A function $f$ is an *even function* if $f(-x) = f(x)$ for every $x$ in the domain of $f$. The graph of an even function is symmetric with respect to the $y$–axis. Functions $f(x) = x^2$, $f(x) = -x^2 + 2x^4$ and $f(x) = \cos x$ are even.

A function $f$ is an *odd function* if $f(-x) = -f(x)$ for every $x$ in the domain of $f$. The graph of an odd function has the origin as point of symmetry, that is, the origin is the midpoint of the segment joining pairs of points of the graph of $f$. Functions $f(x) = x$, $f(x) = x^3 + x^5$ and $f(x) = \sin x$ are odd.

## 1.4 Monotone functions

A function $f$ is said to be monotone *increasing* if for any $x, y \in D(f)$ $x < y$ implies $f(x) \leq f(y)$ (*strictly increasing* if $f(x) < f(y)$).

A function $f$ is said to be monotone *decreasing* if for any $x, y \in D(f)$ $x < y$ implies $f(x) \geq f(y)$ (*strictly decreasing* if $f(x) > f(y)$).

The definitions can be referred to an interval $I$.

**Example 1.4.1.** The function $f(x) = x^2$ is not monotone in $\mathbb{R}$ but it is strictly decreasing in $(-\infty, 0]$ and strictly increasing in $[0, \infty)$. To show that it is strictly increasing in $[0, \infty)$, let $0 \leq x < y$. Then $f(y) - f(x) = y^2 - x^2 = (y - x)(y + x) > 0$ since $y > x \geq 0$.

## 1.5 Monotone functions and the inverse function

**Definition 1.5.1.** A function is bijective when it is injective and suprajective as well.

Obviously, $f : I \to \mathbb{R}$ injective $\iff f : I \to f(I)$ bijective.

**Theorem 1.5.2.** *$f$ has an inverse function on $I \iff f$ injective on $I$.*

**Theorem 1.5.3.** *Let $f$ continuous on $I$. Then:*
$f : I \to \mathbb{R}$ *injective* $\Longleftrightarrow f$ *monotonous on* $I$.

**Theorem 1.5.4.** *Assuming that the inverse function of $f$ exits. Then:*

1. *$f$ increasing $\Longleftrightarrow f^{-1}$ increasing.*
2. *$f$ decreasing $\Longleftrightarrow f^{-1}$ decreasing.*