Universidad Carlos III de Madrid

Exercise	1	2	3	4	Total
Points					

Department of Economics. Introduction to Maths for Economics. Final Exam June 25th 2021

Exam time: 1 hour and 30 minutes.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function** $f(x) = \ln(ex - x^2)$**. Then:**

(a) find the asymptotes of the function and the intervals where $f(x)$ increases and decreases.

(b) find the local and/or global maximum and minimum, and range (or image) of $f(x)$. Draw the graph of the function.

(c) Consider $f_1(x)$ to be the function $f(x)$ defined on the interval where it is increasing, sketch the graph of the inverse function of $f_1(x)$.
(*Hint for the graphs:* use that $1 < \ln 4 < 2$).

0.6 points part a); 0.6 points part b); 0.3 points part c)

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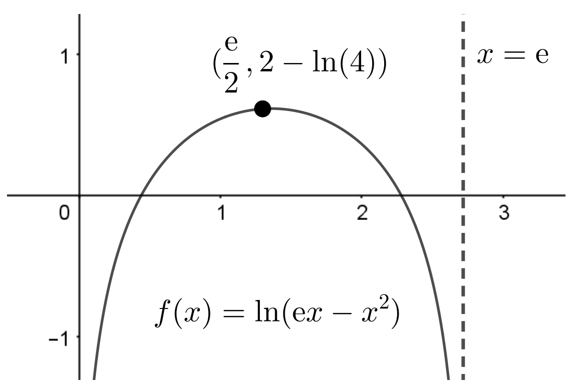
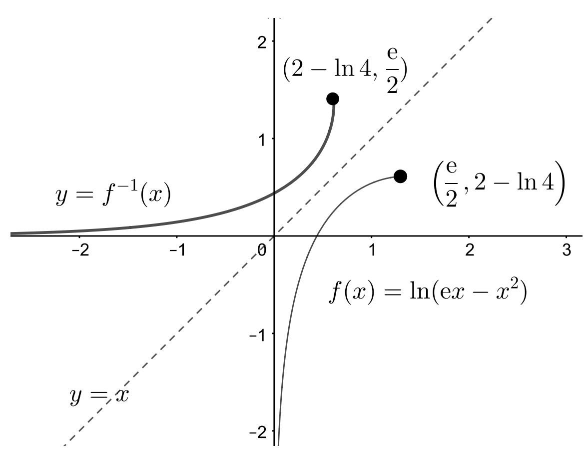
a) The domain of the funcion is $(0, e)$.
Since f is continuous on its domain, a bounded set, it is only needed to study the asymptotes at 0 and e:
i) $\lim_{x \longrightarrow 0^+} f(x) = \ln(0^+) = -\infty$, $\lim_{x \longrightarrow e^-} f(x) = \ln(0^+) = -\infty$.
Therefore $f(x)$ has a vertical asymptote at $x = 0$ and $x = e$.
Moreover, as $f\,'(x) = \dfrac{e - 2x}{ex - x^2}$, we can deduce: f is increasing $\Longleftrightarrow$
$\Longleftrightarrow f\,'(x) > 0 \Longleftrightarrow e - 2x > 0$;
then f is increasing on $(0, \frac{e}{2}]$. Analogously, f is decreasing on $[\frac{e}{2}, e)$.

b) Interpreting the monotonicity of f, it is deduced that there are not local or global minimizers and $x = \frac{e}{2}$ is the only local and global maximizer.
Finally, as $f(\frac{e}{2}) = \ln(e^2/4) = 2 - \ln 4$ and $\lim_{x \longrightarrow 0^+} f(x) = -\infty$ due to the Intermediate Value Theorem we can deduce that the range of the function will be $(-\infty, 2 - \ln 4]$.
The graph of f will have an appearance approximately, similar to the one in figure A.

c) We know that, f_1 is increasing on $(0, \frac{e}{2}]$, $f_1(0^+) = -\infty$, $f_1(\frac{e}{2}) = 2 - \ln 4$. Therefore, the graph of its inverse will be increasing as well, $\lim_{x \longrightarrow -\infty} f_1^{-1}(x) = 0^+$, $f_1^{-1}(2 - \ln 4) = \frac{e}{2}$.
Therefore, the graph of its inverse will have an appearance approximately, similar to the one in figure B:



(A)



(B)

(2) **Given the implicit function $y = f(x)$, defined by the equation $e^{xy} + x^2 + y^2 = 5$ in a neighbourhood of the point $x = 2, y = 0$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 2$.

(b) sketch the graph of the function f near the point $x = 2,\ y = 0$.
Use the tangent line to the graph of $f(x)$ to obtain the approximate values of $f(1.9)$ and $f(2.1)$.

(c) Will $f(2)$ be greater, less or equal than the exact value of $\frac{1}{2}(f(1.9) + f(2.1))$?
(*Hint for part (c):* use that $f''(2) < 0$).
0.6 points part a); 0.5 points part b) 0.4 points part c)

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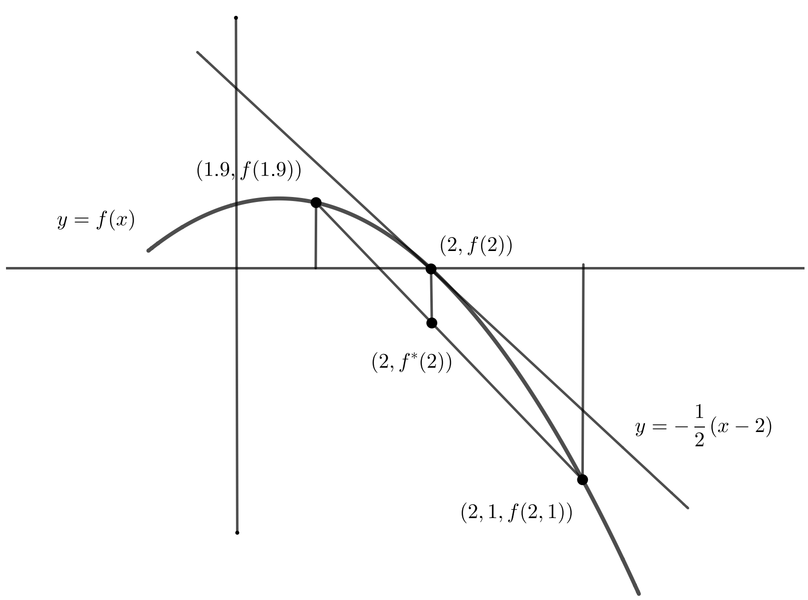
a) First of all, we calculate the first-order derivative of the equation:
$e^{xy}(y + xy') + 2x + 2yy' = 0$
evaluating at $x = 2,\ y = 0$ we obtain: $2y' + 4 = 0 \Longrightarrow y'(2) = f'(2) = -2$.
Then the equation of the tangent line is: $y = P_1(x) = -2(x - 2)$.
Secondly, we calculate the second-order derivative of the equation:
$e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') + 2 + 2(y')^2 + 2yy'' = 0$
evaluating at $x = 2, y = 0, y' = -2$ we obtain $y''(2) = f''(2) = -11$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = -2(x - 2) - \frac{11}{2}(x - 2)^2$

b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 2$ will be as you can see in the figure underneath. On the other hand, using the tangent line, the first order approximation will be:
$f(1.9) \approx -2(-0.1) = 0.2; f(2.1) \approx -2(0.1) = -0.2$.

c) Finally, since $f(x)$ is concave near $x = 2$, $\frac{1}{2}(f(1.9) + f(2.1))$ will be less than $f(2) = 0$, as you can notice looking at the graph below or if you prefer we can calculate its approximate value using the second-order Taylor Polynomial of $f(1.9)$ y $f(2.1)$); $\frac{1}{2}(f(1.9) + f(2.1)) \approx -\frac{11}{2} \cdot 0.1^2$.
Naming $f^*(2) = \frac{1}{2}(f(1.9) + f(2.1))$, the graph will be:



(3) **Let** $C(x) = \sqrt{5x^2 - 6x + 9}$ **be the cost function of a monopolistic firm, where** $x \geq 1$ **represents the quantity in kilograms of the goods. Then:**

(a) find the tangent line of $C(x)$ at $x = 3$, and obtain the approximate value of $C(3.1)$.

(b) **suppose now that** $p(x) = 29 - bx^2$ **is the inverse demand function** with $b \neq 1$, and b very close to 1.
If we know that in the previous period the firm produced 3 units, will the firm encrease or decrease its production in this period?
0.7 points part a); 0.8 points part b)

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a) First of all, we calculate the first-order derivative of the equation: $C'(x) = \dfrac{10x - 6}{2\sqrt{5x^2 - 6x + 9}}$, so
$C'(3) = \dfrac{24}{2\sqrt{36}} = 2.$
Secondly, since $C(3) = 6$, the equation of the tangent line will be:
$y = 6 + 2(x - 3)$
If we approximate the value of $C(3.1)$ using the tangent line, we obtain:
$C(3.1) \approx 6 + 2(3.1 - 3) = 6.2$ monetary units.

b) The profits of the monopolistic firm are:
$B(x) = (29 - bx^2)x - C(x)$ then,
$B'(x) = 29 - 3bx^2 - C'(x) \Longrightarrow B'(3) = 29 - 27b - C'(3) = 27(1 - b).$
Thus, we have two difference cases:
i) if $b < 1$, then the firm would like to increase its production.
ii)but if, $b > 1$ then the firm would reduce its production.

(4) **Let** $f(x) = x^4 - 2x^2$, **you are asked:**

(a) state Bolzano's Theorem (Intermediate Zero Theorem) for the function g defined in the interval $[a, b]$.

(b) suppose that $a = -2$. Find a and b such that $f(x)$ satisfies the hypothesis of this theorem.

(c) suppose that $a = -2$. Find a and b such that $f(x)$ satisfies the thesis of this theorem.

(*Hint for part b) and c):* sketch the graph of the function)

0.3 points part a); 0.5 points part b); 0.7 points part c)

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a) Bolzanos' Theorem states that if g is continuous on the interval $[a, b]$, and the function satisfies that

i) $g(a) < 0 < g(b)$; or

ii) $g(b) < 0 < g(a)$

then there exists a point c in (a, b) such that $g(c) = 0$.

b) since $f(-2) > 0$, then b must satisfy $f(b) < 0$; this is, $b \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$.

c) It is needed a zero of the given polynomial in the interval (a, b).

Thus, $b \in (-\sqrt{2}, \infty)$.

Notice: since $f(x) = x^2(x - \sqrt{2})(x + \sqrt{2})$, we can deduce:

i) $f(x) < 0$ if $x \in (-\sqrt{2}, 0) \cup (0, \sqrt{2})$;

ii) $f(x) > 0$ if $x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$.

Therefore, the graph of f will be approximately: