Universidad Carlos III de Madrid

Department of Economics **Introduction to Mathematics for Economics** **June** $30^{th}, 2020$

Starting time: 9:00 a.m.		**Time: 90 minutes.**
SURNAME:		**NAME:**
ID:	**GRADE:**	**GROUP:**

(1) **Consider the function** $f(x) = x^2 e^{-x}$**. Then:**

(a) Calculate the domain and the asymptotes of the function $f(x)$.

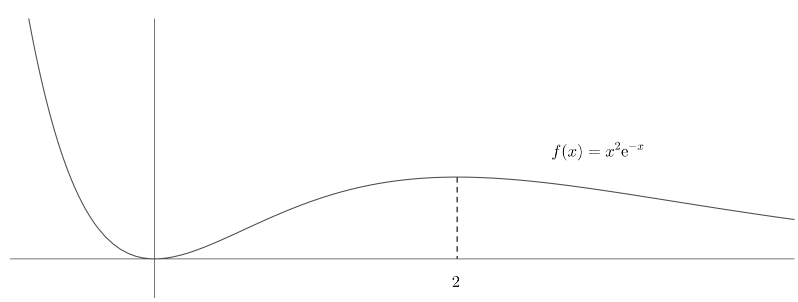
(b) Calculate the intervals where $f(x)$ is increasing, as well as the local and global maxima and minima of $f(x)$. Find the range of $f(x)$ and draw its graph.

Part (a) 0.4 points; Part (b) 0.6 points

---

a) The domain of the function is $\mathbb{R}$.
Since f is continuous in its domain, only asymptotes at ∞ and $-\infty$ are taken into consideration:
i) $\lim_{x \longrightarrow -\infty} \frac{f(x)}{x} = \lim_{x \longrightarrow -\infty} xe^{-x} = -\infty$, thus f has neither horizontal or oblique asymptotes at $-\infty$.
Furthermore, $\lim_{x \longrightarrow -\infty} f(x) = \infty$
ii) $\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$ =[using L'Hopital's rule twice]= $\lim_{x \longrightarrow \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$.
So $f(x)$ has horizontal asymptote $y = 0$ at ∞.

b) Since $f'(x) = e^{-x}(-x^2 + 2x)$, we deduce that: f is increasing $\Longleftrightarrow f'(x) > 0 \Longleftrightarrow -x^2 + 2x = x(-x+2) > 0$;
so f is increasing in $[0, 2]$. Analogously, f is decreasing in $(-\infty, 0]$ y en $[2, \infty)$. Since f is never negative and $\lim_{x \longrightarrow -\infty} f(x) = \infty$, we can deduce that 2 is a local maximizer and 0 is a local and global minimizer.
Because there are no other critical points, there cannot be any other local or global minimizer.
Finally, since $\lim_{x \longrightarrow -\infty} f(x) = \infty$, and the intermediate value theorem we can deduce that the range is $[0, \infty)$.
We conclude that the graph of f is represented approximately like this:



(2) **Given the function $y = f(x)$, defined implicitly by the equation $xe^{-y} + 2y = 1$ around the point $x = 1, y = 0$, answer the following questions:**

(a) find the tangent line and the second order Taylor polynomial of the implicit function at the point $a = 1$.

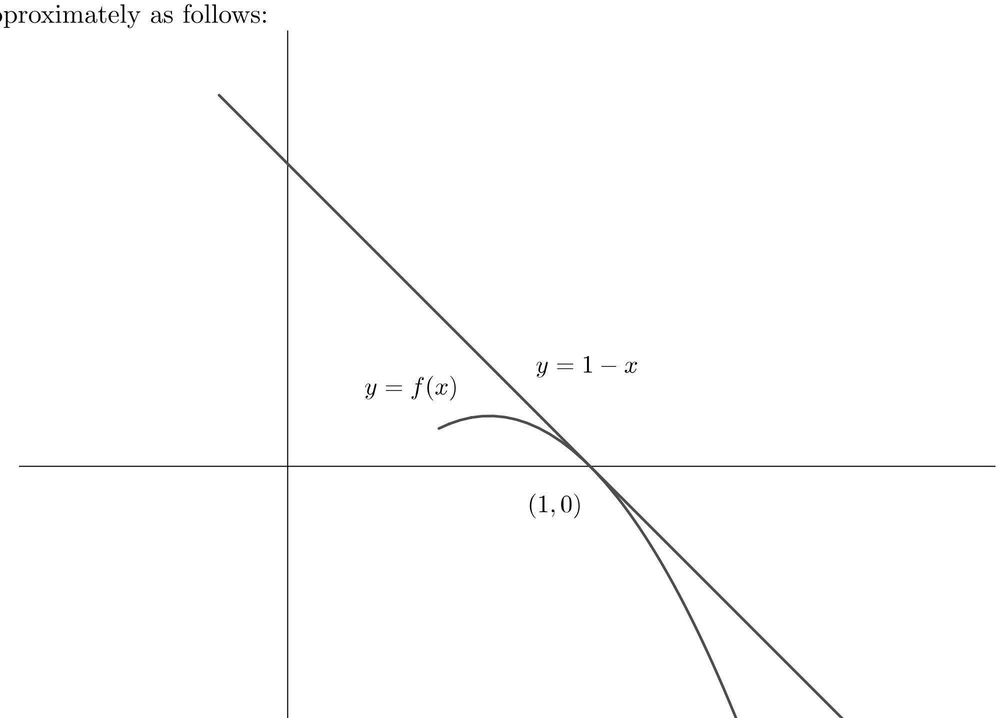
(b) draw the graph of f around the point $x = 1$ and using the tangent line, obtain the approximated values of $f(0.9)$ and $f(1.2)$.
Justify if some of the above approximations are by excess or by default.
Part (a) 0.6 points; Part (b) 0.4 points

---

a) Firstly, we calculate the first order derivative of the function: $(1 - xy')e^{-y} + 2y' = 0$
evaluating at $x = 1,\ y(1) = 0$ we can deduce:
$1 - y' + 2y' = 0 \Longrightarrow y'(1) = f'(1) = -1$.
So, the tangent line is: $y = P_1(x) = 0 - (x - 1)$ or $y = 1 - x$.

Analogously, we calculate the second order derivative of the function:
$[-y' - xy'' + (1 - xy')(-y')]e^{-y} + 2y'' = 0$.
evaluating at $x = 1, \quad y(1) = 0, \quad y'(1) = -1$ we know that:
$[1 - y'' + 2] + 2y'' = 0 \Longrightarrow y''(1) = f\,''(1) = -3$.
So, the second order Taylor polynomial is: $y = P_2(x) = 1 - x - \frac{3}{2}(x - 1)^2$.

b) Using the second order Taylor polynomial, the graph of the function f around the point $x = 1$ will be approximately as follows:



On the other hand, the first order approximations will be:
$f(0.9) \approx P_1(0.9) = 1 - (0.9) = 0.1$; $f(1.2) \approx P_1(1.2) = 1 - (1.2) = -0.2$.
Since the function is concave around $x = 1$, because $f''(1) < 0$, the approximation of the values of f using the tangent line will be in both cases by excess.

(3) **Let $C(x) = 100 + 150x + x^2$ be the cost function and $p(x) = 250 - x$ be the inverse demand function of a monopolist firm, with $0 \leq x \leq 50$ being the number of units produced of a given good then:**

(a) find the price p^* and the production level x^* which maximize the profits of the firm.

(b) Suppose that the government increases the costs of production by a tax of T euros per unit produced. Find the new production level $x^*(T)$ and the new price $p^*(T)$ which maximize the profit of the company.

Compare the results obtained with those obtained in part (a) above.

Part (a) 0.5 points; Part (b) 0.5 points

---

a) First of all, we calculate the profit function.
$B(x) = (250 - x)x - (100 + 150x + x^2) = -2x^2 + 100x - 100.$
Then, we calculate the first and second order derivatives of B:
$B'(x) = -4x + 100$; $B''(x) = -4 < 0$
we can see that B has only one critical point at $x^* = \dfrac{100}{4} = 25$ and since B is a concave function, that point is the only global maximizer.
Finally, $p^* = p(25) = 250 - 25 = 225.$

b) Since, the new cost function becomes $C(x) = 100 + (150 + T)x + x^2$,
the new profit function is $B(x) = -2x^2 + (100 - T)x + 100.$
we calculate the first and second order derivatives of B:
$B'(x) = -4x + 100 - T \quad B''(x) = -4 < 0,$
then we see that B has only one critical point at $x^*(T) = \dfrac{100 - T}{4} = 25 - \dfrac{T}{4}.$
Since B is a concave function, the critical point is the only global maximizer.
Finally, $p^*(T) = 250 - (25 - \dfrac{T}{4}) = 225 + \dfrac{T}{4}.$
In comparison with the case without taxes ($T = 0$), the output has decreased and the price has increased.

(4) **Let the function** $f(x) = \begin{cases} ax+3 & si\ x<2 \\ -x^2+2ax+b & si\ \ x \geqslant 2 \end{cases}$ **and let us consider the interval** $[0,4]$ **.**

Then:

(a) Find a and b in order that $f(x)$ satisfies the hypothesis (or initial conditions) of Lagrange's theorem (or mean value theorem) in that interval.

(b) For those values a, b find the intermediate value or values c in such a way that the thesis (or conclusion) of this theorem is satisfied.

Hint for both parts: state the mean value theorem.

Part (a) 0.5 points; Part (b) 0.5 points

---

a) We need to set the continuity and derivability at $x=2$.

For that reason, as $\lim_{x \longrightarrow 2^-} f(x) = 2a+3, f(2) = \lim_{x \longrightarrow 2^+} f(x) = -4+4a+b$ it can be deduced that the function will be continuous at that point when:

$2a+3 = -4+4a+b \Longleftrightarrow 2a+b=7.$

On the other hand, supposing the the function is continuous at $x=2$, it will have a derivative at that point if: $a = f'_-(2) = f'_+(2) = -4+2a \Longleftrightarrow a=4.$

So the function will be continuous and derivable at $x=2$ when $a=4, b=-1$.

b) By the mean value theorem we know that:

There exists $c \in (0,4) : f(4) - f(0) = f'(c)(4-0).$

Taking into account that $a=4, b=-1$, the former equation is equivalent to $(-16+32-1)-(3) = 12 = 4f'(c)$.

In other words: $f'(c) = 3.$

When $x \leq 2$, $f'(x) = a = 4 \neq 3$, so it is not possible that $c \leq 2$.

When $x > 2$, $f'(x) = -2x+2a = -2x+8 = 3 \Longleftrightarrow x = \frac{5}{2}.$

So the only possible value is $c = \frac{5}{2}$.