1

Consider the function $f(x) = \dfrac{\ln(1+x)}{\sqrt{1+x}}$. Then:

(a) (5 points) Find its domain and its asymptotes.

(b) (10 points) find the intervals where $f(x)$ increases and decreases, its local and global extrema and range (or image). Draw the graph of the function.

Solution:

(a) The domain of the given function is $(-1, \infty)$. Therefore, there isn't an asymptote at $-\infty$.

Since the function is continuous on its domain we only need to study its possible vertical asymptotes at -1^+ :

$\lim_{x \longrightarrow -1^+} f(x) = \dfrac{\ln(0^+)}{0^+} = \dfrac{-\infty}{0^+} = -\infty$; thus, the function has a vertical asymptote at $x = -1^+$.

Since $\lim_{x \longrightarrow \infty} f(x) = \lim_{x \longrightarrow \infty} \dfrac{\ln(1+x)}{\sqrt{1+x}} = \dfrac{\infty}{\infty} =$ (applying L'Hopital) $=$

$= \lim_{x \longrightarrow \infty} \dfrac{1/(1+x)}{1/2\sqrt{1+x}} = \lim_{x \longrightarrow \infty} \dfrac{2}{\sqrt{1+x}} = 0.$

we know that the function has a horizontal asymptote $y = 0$ at ∞.

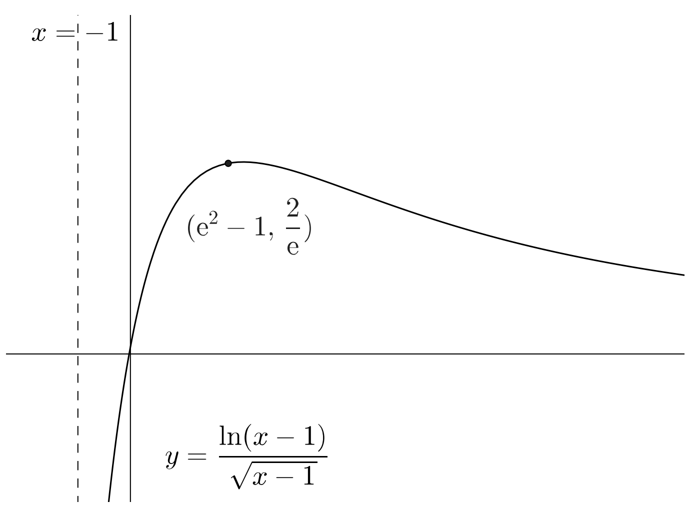
(b) Since $f'(x) = \dfrac{(1+x)^{-1}\sqrt{1+x} - (1/2\sqrt{1+x})\ln(1+x)}{1+x}$, we can deduce that:

f is increasing $\Longleftrightarrow f'(x) > 0 \Longleftrightarrow (1+x)^{-1}\sqrt{1+x} - (1/2\sqrt{1+x})\ln(1+x) > 0 \Longleftrightarrow$ (multiplying by

$2\sqrt{1+x}) \Longleftrightarrow 2 - \ln(1+x) > 0 \Longleftrightarrow 2 > \ln(1+x) \Longleftrightarrow 1+x < e^2$; so, f is increasing on $(-1, e^2-1]$. Analogously, f is decreasing on $[e^2-1, \infty)$. We can deduce that f attains a local and global maximum at $x = e^2 - 1$ but it hasn't got a global or a local minimum.

Since $f(e^2-1) = \dfrac{\ln(e^2)}{\sqrt{e^2}} = \dfrac{2}{e}$, $\lim_{x \longrightarrow 0^+} f(x) = -\infty$ and $\lim_{x \longrightarrow \infty} f(x) = 0$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $(-\infty, f(e^2-1)] = (-\infty, \dfrac{2}{e}]$.

Thus, the graph of f will have an appearance approximately similar to:



2

(a) (8 points) Show that the equation $\ln(x+2y) - y = 0$ defines an implicit function $y = f(x)$ around the point $x = 1, y = 0$. Find $f'(1)$ and $f''(1)$.

(b) (7 points) Find both the tangent line to the graph of $f(x)$ at $x = 1$ and the Taylor polynomial of $f(x)$ of order 2 at $x = 1$. Sketch the graph of the function f next to the point $x = 1$.

Solution:

(a) The point $x = 1$, $y = 0$ satisfies the equation, since $\ln(1 + 2\cdot 0) - 0 = \ln 1 = 0$; the equation is defined by the function $F(x,y) = \ln(x+2y) + 3y$, which of class C^2 around the point and $\dfrac{\partial F}{\partial y} = \dfrac{2}{x+2y} - 1$, which evaluated at $(1,0)$ has the value $1 \neq 0$. Thus, the conditions imposed in the Implicit Function Theorem are fulfilled, and the equation defines a unique implicit function $y = f(x)$ in a certain interval centered at $x = 1$, such that $f(1) = 0$. To find $f'(1)$, we derive the equation with respect to x and substitute the values $x = 1$ and $y = 0$:

$$\frac{1+2y'}{x+2y} - y' = 0 \quad \Rightarrow \quad 1 + 2f'(1) - f'(1) = 0 \quad \Rightarrow f'(1) = -1.$$

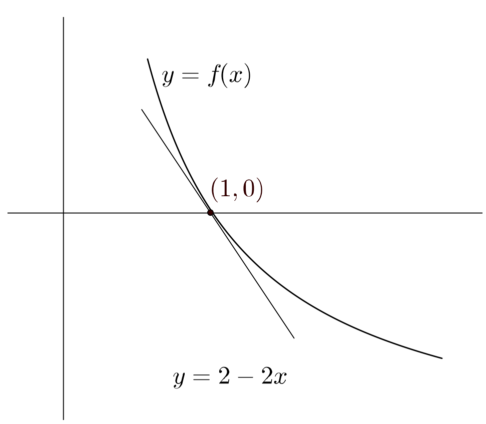
To calculate $f''(1)$, derive again in $\dfrac{1+2y'}{x+2y} - y' = 0$ with respect to x, and substitute the values $x = 1$, $y = 0$ and $y' = -1$, to obtain

$$\frac{2y''(x+2y) - (1+2y')^2}{(x+2y)^2} - y'' = 0 \quad \Rightarrow \quad 2y'' - (1-2)^2 - y'' = 0 \quad \Rightarrow f''(1) = 1.$$

(b) By part (a), the tangent line of the implicit function is given by $y - f(1) = f'(1)(x-1) = -(x-1)$. The second order Taylor polynomial is

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = -(x-1) + \frac{1}{2}(x-1)^2.$$

The graph of the implicit function near $x = 1$ is depicted below



3

Let $C'(x) = a + 10x$ be the marginal cost function and $C_0 = 80$ the fixed cost of a monopolistic firm, being $x \geqslant 0$ the number of units produced of certain goods. Then:

(a) (8 points) Calculate the production x_0 such that minimizes the average cost of the firm.
Hint: the production x_0 could depend on a or not.

(b) (7 points) Suppose now that $p(x) = 100 - 5x$ is the inverse demand function of the firm, calculate a such that the maximum profit is attained at the production $x = 4$.

Solution:

(a) First of all, we calculate the total cost function $C(x) = 80 + ax + 5x^2$

Secondly, the average cost function is $\dfrac{C(x)}{x} = \dfrac{80}{x} + a + 5x$

Now, we calculate the critical points of this function:

$\left(\dfrac{C(x)}{x}\right)' = -\dfrac{80}{x^2} + 5 = 0 \Longleftrightarrow x^2 = 16 \Longleftrightarrow x = 4;$

Since the average cost function is convex, because $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{160}{x^3} > 0,$

we can deduce that $x = 4$ is the unique global minimizer of the average cost function.

Notice that a can take any value.

(b) The profit function is:

$B(x) = (100 - 5x)x - (80 + ax + 5x^2) = -10x^2 + (100 - a)x - 80.$

we calculate its first and second order derivatives:

$B'(x) = -20x + 100 - a;\ B''(x) = -20 < 0$

we see that $B'(x) = 0 \Longleftrightarrow x = \dfrac{100 - a}{20} = 4 \Longleftrightarrow a = 20$. Thus, $x = 4$ is the only critical point of the function $B(x)$.

Since the profit function is concave this critical point is the unique global maximizer when $a = 20$.

4

Given the function $f(x) = xe^{2-x}$, answer the following questions.

(a) (5 points) Find the inflection points of $f(x)$ and its convexity and concavity intervals.

(b) (10 points) Study the global extrema of $f(x)$ in the interval $[0, 3]$.
Hint for (b): Use that $2 < e < 3$, or study the monotonicity of f.

Solution:

(a) We calculate the first and second order derivatives of the function $f(x)$:
$f'(x) = e^{2-x} - xe^{2-x} = (1-x)e^{2-x}$;
$f''(x) = -e^{2-x} - (1-x)e^{2-x} = (x-2)e^{2-x}$
So, f is concave if $x < 2$, since $f''(x) < 0$
and f is convex if $x > 2$, because $f''(x) > 0$
Therefore, $x = 2$ is the only inflection point of the function.

(b) The only point that vanishes the derivative is $x = 1$, by part (a). Thus, since Weierstrass Theorem holds for this function and interval, the global extrema are 0, 1 or 2. Evaluating f we find the maximum and the minimum: $f(0) = 0$, $f(1) = e$ and $f(3) = \frac{3}{e}$. Clearly, $x = 0$ is the global minimum and $x = 1$ is the global maximum, since

$$f(0) = 0 < f(3) = \frac{3}{e} < 1 < e = f(1).$$

Another way: as f is increasing for $x < 1$ and decreasing for $x > 1$, then $x = 1$ is a global maximum (even in the whole real line). Since $f(0) = 0 < f(3) = \frac{3}{e}$, then f has its global minimum in $[0, 3]$ at $x = 0$.