Universidad Carlos III de Madrid

Exercise	1	2	3	4	Total
Points					

Department of Economics **Introduction to Mathematics for Economics**

January 26th 2021, Final Exam. Exam time: 2 hours.

LAST NAME:		**FIRST NAME:**
ID:	**DEGREE:**	**GROUP:**

(1) **Consider the function $f(x) = (x+1)^2 e^{-x}$. Then:**

(a) find the asymptotes of the function and the intervals where $f(x)$ increases and decreases.

(b) find the global maximum and minimum, and range (or image) of $f(x)$. Draw the graph of the function.

(c) consider $f_1(x)$ to be the function $f(x)$ defined on the interval $[-1, 1]$, sketch the graph of the inverse function of $f_1(x)$.

(*Hint for part (c):* do not try to calculate the explicit formula of the inverse function of f_1)

0.6 points part a); 0.6 points part b); 0.3 points part c)

(a) The domain of the function is $\mathbb{R}$.

Since f is continuous on its domain, we only need to study its asymptotes at ∞ and $-\infty$:

i) $\lim_{x\longrightarrow\infty} f(x) = \lim_{x\longrightarrow\infty} \frac{(x+1)^2}{e^x} = \frac{\infty}{\infty} =$ [applying L'Hopital's Rule twice] $= \lim_{x\longrightarrow\infty} \frac{2}{e^x} =$
$= \frac{2}{\infty} = 0$. Therefore $f(x)$ has a horizontal asymptote $y = 0$ at ∞.

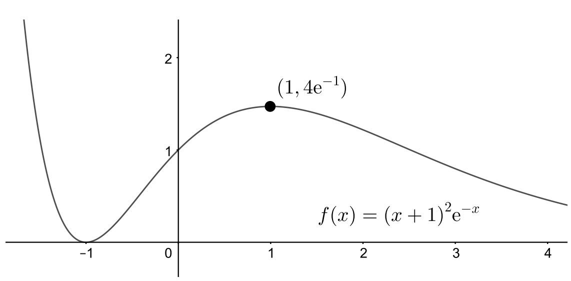
ii) $\lim_{x\longrightarrow-\infty} \frac{f(x)}{x} = \lim_{x\longrightarrow-\infty} \frac{(x+1)^2}{x} . \lim_{x\longrightarrow-\infty} e^{-x} = -\infty$, then f has no horizontal neither oblique asymptote at $-\infty$.

As $f'(x) = e^{-x}(1-x^2)$, we can deduce: f is increasing $\Longleftrightarrow f'(x) > 0 \Longleftrightarrow 1 - x^2 > 0$; then f is increasing on $[-1, 1]$. Analogously, f is decreasing on $(-\infty, -1]$ and $[1, \infty)$.

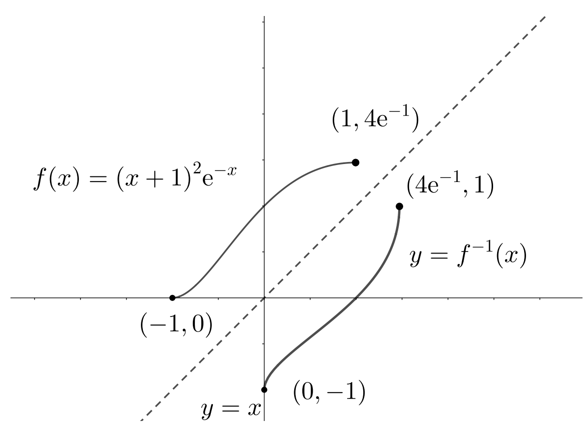
(b) Interpreting the monotonicity of f, it is deduced that -1 is a local minimizer and 1 is a local maximizer. Since $\lim_{x\longrightarrow-\infty} f(x) = \infty$, there is no global maximum. In addition, as $f(-1) = 0$ and $f(x) > 0$ (if $x \neq -1$), it is deduced that 0 is a strict (unique) global minimizer. Finally, as $f(-1) = 0, f(x) \geqslant 0$ and $\lim_{x\longrightarrow-\infty} f(x) = \infty$, due to the Intermediate Value Theorem we can deduce that the range of the function will be $[0, \infty)$.

The graph of f will have an appearance approximately, similar to the one in figure A.

(c) We know that, f_1 is increasing on $[-1, 1]$, $f_1(-1) = 0$, $f_1(1) = 4/e$. Therefore, the graph of its inverse will have an appearance approximately, similar to the one in figure B:



(A)



(B)

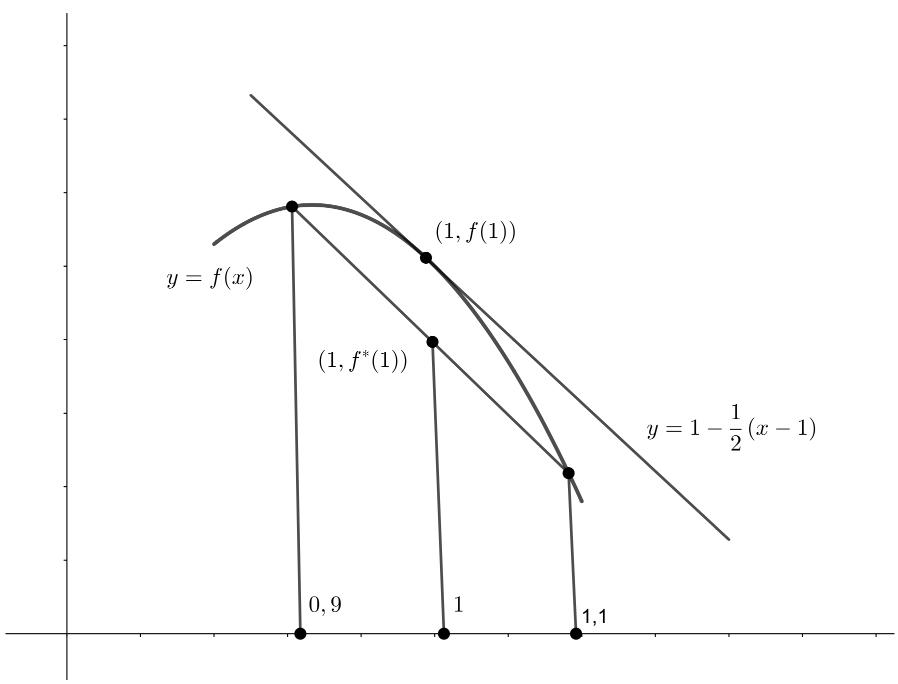
(2) **Given the implicit function $y = f(x)$, defined by the equation $e^x + ye^y = 2e$ in a neighbourhood of the point $x = 1, y = 1$, it is asked:**

(a) find the tangent line and the second-order Taylor Polynomial of the function at $a = 1$.

(b) sketch the graph of the function f near the point $x = 1, y = 1$. Use the tangent line to the graph of $f(x)$ to obtain the approximate values of $f(0.9)$ and $f(1.1)$.
Will $f(1)$ be greater, less or equal than the exact value of $\frac{1}{2}(f(0.9) + f(1.1))$?
(*Hint for part (b):* use that $f''(1) < 0$.
0.8 points part a); 0.7 points part b)

(a) First of all, we calculate the first-order derivative of the equation:
$e^x + y'e^y + yy'e^y = e^x + y'(y+1)e^y = 0$
evaluating at $x = 1$, $y(1) = 1$ we obtain: $y'(1) = f'(1) = -1/2$.
Then the equation of the tangent line is: $y = P_1(x) = 1 - \frac{1}{2}(x-1)$. Secondly, we calculate the second-order derivative of the equation:
$e^x + y''(y+1)e^y + (y')^2e^y + y'(y+1)y'e^y = 0$
evaluating at $x = 1$, $y(1) = 1$, $y'(1) = -1/2$ we obtain $y''(1) = f''(1) = -7/8$.
Therefore, the second-order Taylor Polynomial is: $y = P_2(x) = 1 - \frac{1}{2}(x-1) - \frac{7}{16}(x-1)^2$.

(b) Using the second-order Taylor Polynomial, the approximate graph of the function f, near the point $x = 1$, will be as you can see in the figure underneath. On the other hand, using the tangent line, the first order approximation will be:
$f(1.1) \approx 1 - \frac{1}{2}(0.1) = 0.95$; $f(0.9) \approx 1 - \frac{1}{2}(-0.1) = 1.05$.
Finally, since $f(x)$ is concave, $\frac{1}{2}(f(0.9) + f(1.1))$ will be less than $f(1)$, as you can notice looking at the graph below or if you prefer we can calculate its approximate value using the second-order Taylor Polynomial: $\frac{1}{2}(f(0.9) + f(1.1)) \approx 1 - \frac{7}{8}0.01 < f(1) = 1$.
Naming $f^*(1) = \frac{1}{2}(f(0.9) + f(1.1))$, the graph will be:



(3) **Let** $C(x) = C_0 + 50x + \frac{1}{2}x^2$ **be the cost function and** $p(x) = 710 - 5x$ **the inverse demand function of a monopolistic firm. Then:**

(a) calculate the price p^* and the production x^* that maximizes the profit.

(b) find C_0 such that the production obtained in part a) would be the same that minimizes the average cost.

0.6 points part a); 0.9 points part b)

(a) First of all, we calculate the profit function.
$B(x) = (710 - 5x)x - (C_0 + 50x + \frac{1}{2}x^2) = -\frac{11}{2}x^2 + 660x - C_0$
Secondly, we calculate the first and second order derivatives of B:
$B'(x) = -11x + 660;\ B''(x) = -11 < 0$
we see that B has a unique critical point at $x^* = \dfrac{660}{11} = 60$ and, since B is a concave function, the critical point is the unique global minimizer.
Finally, $p^* = p(60) = 710 - 300 = 410$

(b) The average cost function is $\dfrac{C(x)}{x} = \dfrac{C_0}{x} + 50 + \frac{1}{2}x$,
its first order derivative: $\left(\dfrac{C(x)}{x}\right)' = -\dfrac{C_0}{x^2} + \frac{1}{2} = 0 \Longleftrightarrow x^2 = 2C_0$.
Since $\left(\dfrac{C(x)}{x}\right)'' = \dfrac{2C_0}{x^3} > 0$, the function is convex and the critical point will be the global minimizer.
Since $x^* = 60$ must be the minimizer, the solution will be
$60 = x^* = \sqrt{2C_0} \Longrightarrow C_0 = 1800$.

(4) **Let** $f(x) = \begin{cases} (x+a)^2, & x < 2 \\ b, & x = 2 \\ -x^2+6x+1, & x > 2 \end{cases}$ be a piece-wise defined function in the interval $[1,3]$. **Then:**

(a) state Weierstrass' Theorem for a function g defined in an interval I. Calculate a y b such that $f(x)$ satisfies the hypothesis of this theorem.

(b) suppose that $a = -1$, find the values of b such that the thesis (or conclusion) of Weierstrass' Theorem is satisfied in the interval $[1,3]$. What can you say for the intervals $[1,2]$ or $[2,3]$?

0.6 points part a); 0.9 points part b)

(a) The hypothesis is that g is continuous in an interval I closed and bounded. The thesis (or conclusion) is that the function g attains its global maximum and minimum on I.

Thus, we need that the function f is continuous at $x = 2$.

Since, $\lim\limits_{x \longrightarrow 2^+} f(x) = -4 + 12 + 1 = b = f(2) \Longrightarrow b = 9$.

And $\lim\limits_{x \longrightarrow 2^-} f(x) = (2+a)^2 = 9 = f(2) \Longrightarrow a = -5$ or $a = 1$.

Therefore, we can deduced that the function will be continuous in $[1,3]$ when: $b = 9$ and ($a = -5$ or $a = 1$).

(b) For the value $a = -1$ the hypothesis of the theorem is not satisfied in the interval $[1,3]$. Meanwhile, it could be possible that the thesis is satisfied in this interval depending on the values of b.

If we notice that f is increasing in $[1,2)$ and also in $(2,3]$, and furthermore:

$0 = f(1) < \lim\limits_{x \longrightarrow 2^-} f(x) = 1 < 9 = \lim\limits_{x \longrightarrow 2^+} f(x) < f(3) = 10$.

We can consider three different cases depending on b:

i) $b \leq 0 \Longrightarrow \min f = b, \max f = 10$.

ii) $0 \leq b \leq 10 \Longrightarrow \min f = 0, \max f = 10$.

iii) $10 \leq b \Longrightarrow \min f = 0, \max f = b$.

Then, for any real value of b the thesis of Weierstrass' Theorem is satisfied.

Now, in the case of the interval $[1,2]$ the theorem is only satisfied if $b \geqslant 1$, and it happens that $\min f = 0, \max f = b$. Notice that if $b < 1$ the maximum doesn't exist as we can appreciate in the left graph below.

Analogously, in the case of the interval $[2,3]$ the theorem is only satisfied if $b \leq 9$, and it happens that $\min f = b$, $\max f = 10$. Notice that if $b > 9$ the minimum doesn't exist, as we can appreciate in the right graph below.