1

Consider the function $f(x) = x^3 e^{-x}$.

(a) (5 points) Calculate the domain and the asymptotes of the function f.

(b) (10 points) Calculate the intervals where f is increasing or decreasing, as well as the local and global maxima and minima of f. Find the image of f and draw its graph.

Solution:

(a) Since that f is continuous in its domain, only asymptotes at ∞ and $-\infty$ are taken into consideration:

i) $\lim\limits_{x\longrightarrow-\infty} \frac{f(x)}{x} = \lim\limits_{x\longrightarrow-\infty} x^2e^{-x} = \infty$, thus f has neither horizontal nor oblique asymptotes at $-\infty$.

ii) $\lim\limits_{x\longrightarrow\infty} f(x) = \lim\limits_{x\longrightarrow\infty} \frac{x^3}{e^x} = \frac{\infty}{\infty} =$ [after three applications of L'Hopital rule] $= \lim\limits_{x\longrightarrow\infty} \frac{6}{e^x} = \frac{6}{\infty} = 0$.

Hence, $f(x)$ has horizontal asymptote $y = 0$ at ∞.

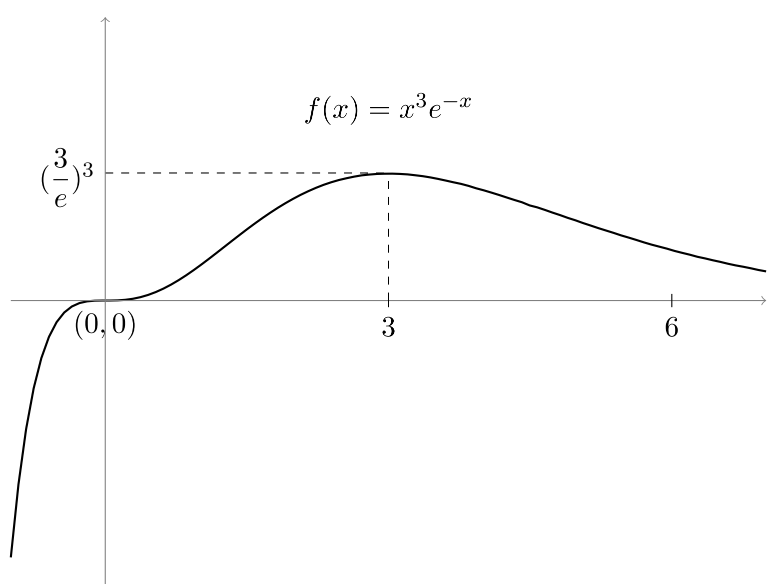
(b) Since $f'(x) = e^{-x}(-x^3 + 3x^2)$, we deduce that:

f is increasing $\Leftrightarrow f'(x) > 0 \Leftrightarrow -x^3 + 3x^2 = x^2(-x + 3) > 0$; So f increasing in $(-\infty, 3]$. In the same way, f is decreasing in $[3, \infty)$.

From this study we conclude that 3 is both a local and global maximizer. Since there are no other critical points, there cannot be minimizers.

Finally, since $\lim\limits_{x\longrightarrow-\infty} f(x) = -\infty$ and that a continuous function satisfies the Theorem of the Intermediate Values, we get that the image of f is $(-\infty, f(3)] = (-\infty, 27e^{-3}]$.

The figure below shows the graph of f.



2

Given the function $y = f(x)$ defined implicitly by the equation

$$xe^y + ye^x = 2e$$

around the point $x = 1, y = 1$, answer the following questions:

(a) (8 points) Find the tangent line and the Taylor polynomial of degree two of the implicit function at the point $a = 1$.

(b) (7 points) Draw the graph of f around the point $x = 1$ and using the tangent line, obtain approximated values of $f(0.9)$ and $f(1.2)$.
Justify if some of the above approximations are by excess or by default.

Solution:

(a) Differentiating once in the equation we get:

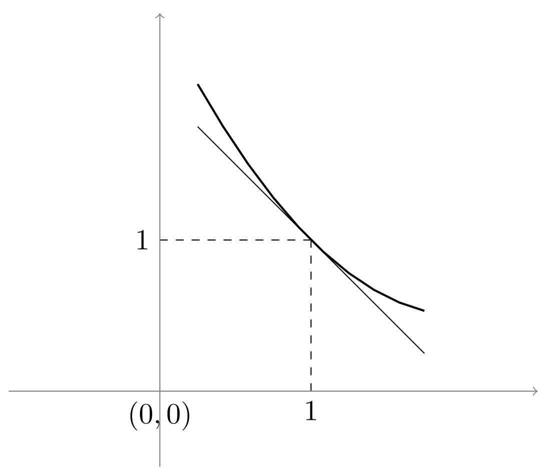
$$(1 + xy')e^y + (y' + y)e^x = 0$$

and plugging into $x = 1,\ y(1) = 1$, we have $2e(y' + 1) = 0 \Longrightarrow y'(1) = f'(1) = -1$.
The equation of the tangent line is: $y = P_1(x) = 1 - (x - 1)$; that is, $y = 2 - x$.
Differentiating again we get:

$$(y' + xy" + (1 + xy')y')e^y + (y" + 2y' + y)e^x = 0$$

and plugging into $x = 1, y(1) = 1, y'(1) = -1$ we have $2e(y" - 1) = 0 \Rightarrow y"(1) = f"(1) = 1$.
The second order Taylor polynomial is: $y = P_2(x) = 1 - (x - 1) + \frac{1}{2}(x - 1)^2$.

(b) The second order Taylor polynomial approximates the graph of f near to the point $x = 1$ and it is shown in the figure below.



On the other hand the approximated values of first order are:

$$f(0.9) \approx P_1(0.9) = 1 - (-0,1) = 1,1, \qquad f(1,2) \approx P_1(1.2) = 1 - (0.2) = 0.8.$$

Since the function is convex near $x = 1$, due to $f"(1) > 0$, the approximated values obtained with the tangent line are by default in both cases.

3

Let $C(x) = 85 + 100x - x^2$ be the cost function and $p(x) = 200 - 3x$ be the inverse demand function of a monopolist firm, with $0 \leq x \leq 50$ being the number of units produced of a given good. Answer the following questions:

(a) (6 points) Find the price p^* and production level x^* which maximize the profits of the firm.

(b) (9 points) Suppose that the government diminishes production cost by means of a subsidy of S euros per unit produced. Find the new production level $x^*(S)$ and the new price $p^*(S)$ which maximize the profit of the company.

Compare the results obtained with those obtained in part (a) above.

Solution:

(a) Profits:

$$B(x) = (200 - 3x)x - (85 + 100x - x^2) = -2x^2 + 100x - 85.$$

The first and the second derivatives are: $B'(x) = -4x + 100; B"(x) = -4 < 0$, hence B has $x^* = \dfrac{10}{4} = 25$ as its unique critical point, which is the unique global maximizer of B since this function is strictly concave.

Finally, $p^* = p(25) = 200 - 75 = 125$.

(b) The cost function becomes $C(x) = 85 + (100 - S)x - x^2$, and consequently the new profit function is $B(x) = -2x^2 + (100 + S)x - 85$. The first and the second derivatives are: $B'(x) = -4x + 100 + S$ and $B"(x) = -4 < 0$, hence B has $x^*(S) = \dfrac{100 + S}{4} = 25 + \dfrac{S}{4}$ as its unique critical point, which is the unique global maximizer of B since this function is strictly concave.

Finally, $p^*(S) = 200 - 3\left(25 + \dfrac{S}{4}\right) = 125 - 3\dfrac{S}{4}$.

In comparison with the case without subsidies ($S = 0$), the output has increased and the price has decreased, for every $S > 0$.

4

Given the function $f(x) = x^2 \ln x$, answer the following questions:

(a) (8 points) Find the intervals where the function f is increasing or decreasing, and calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) (7 points) Discuss whether f attains global maximum and/or global minimum in the interval $(0, b]$, where $b > 0$ is a parameter.
Hint: draw the graph of f.

Solution:

(a) The first derivative is

$$f'(x) = 2x \ln x + x^2 (\frac{1}{x}) = x(2 \ln x + 1).$$

Since that the domain of f is $(0, \infty)$, we can infer:

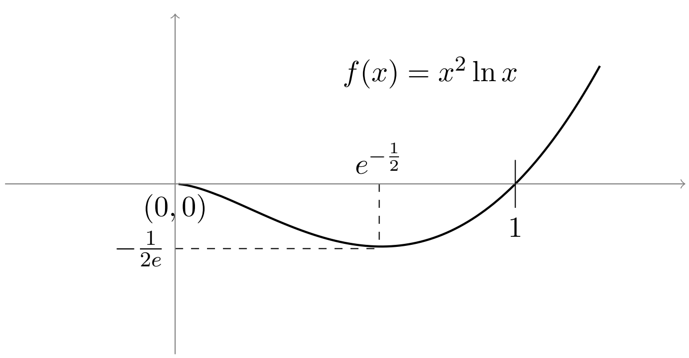
f is increasing if $2 \ln x + 1 > 0 \iff \ln x > -\frac{1}{2} \iff x \geq e^{-1/2}$; and

f is decreasing if $2 \ln x + 1 < 0 \iff \ln x < -\frac{1}{2} \iff 0 < x \leq e^{-1/2}$.

Thus, $x = e^{-1/2}$ is the unique global minimizer in $(0, \infty)$.

On the other hand, $\lim\limits_{x\longrightarrow 0^+} f(x) = \lim\limits_{x\longrightarrow 0^+} \frac{\ln x}{1/x^2} = \frac{-\infty}{\infty}$ =[L'Hopital]= $= \lim\limits_{x\longrightarrow 0^+} \frac{1/x}{-2/x^3} = 0$; finally: $\lim\limits_{x\longrightarrow\infty} f(x) = \infty$

(b) The figure below shows the graph of f.



We observe the following facts:

i) when $0 < b \leq e^{-1/2}$, f is decreasing in $(0, b]$.

As a consequence, b is the global minimizer of f but there is no global maximizer.

ii) when $e^{-1/2} < b < 1$, f is decreasing in $(0, e^{-1/2}]$ and increasing in $[e^{-1/2}, b]$; but $f(b) < f(1) = 0$ since $b < 1$, thus f does not have global maximum in $(0, b]$, given that $\lim\limits_{x\longrightarrow 0^+} f(x) = 0$.

Obviously, the global minimum is attained at $x = e^{-1/2}$.

iii) when $1 \leq b$, f is decreasing in $(0, e^{-1/2}]$ and increasing in $[e^{-1/2}, b]$; but $0 = f(1) \leq f(b)$ since $1 \leq b$, thus f attains the global maximum at $x = b$, given that $\lim\limits_{x\longrightarrow 0^+} f(x) = 0$.

Obviously, the global minimum is attained at $x = e^{-1/2}$.