1

Consider the function $f(x) = \sqrt{x^2 - x}$.

(a) (15 points) Determine the domain and calculate the asymptotes of f. Determine the image of f.

(b) (15 points) Find the intervals where f is monotone, as well as the intervals where f is concave or convex. Draw the graph of f.

Solution:

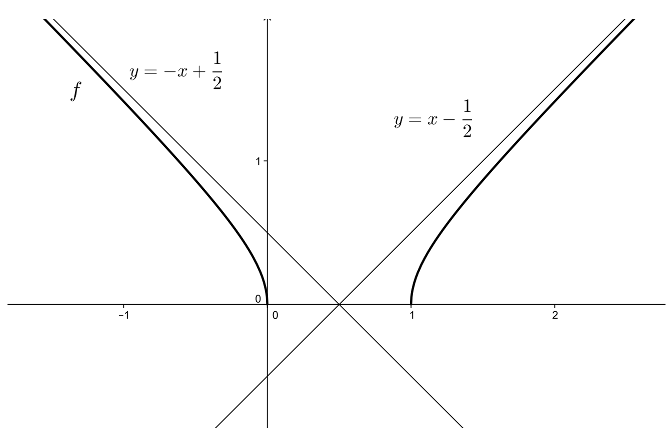
(a) The domain of f is the set $\{x : x^2 - x = x(x-1) \geq 0\} = (-\infty, 0] \cup [1, \infty)$. Regarding the asymptotes, note that f is continuous in its domain, and that the domain is the union of closed intervals, thus there are no vertical asymptotes. On the other hand

- $\lim\limits_{x \longrightarrow \pm\infty} \dfrac{f(x)}{x} = \lim\limits_{x \longrightarrow -\infty} \dfrac{\sqrt{x^2 - x}}{\pm\sqrt{x^2}} = \pm \lim\limits_{x \longrightarrow -\infty} \sqrt{1 - \frac{1}{x}} = \pm 1.$
- $\lim\limits_{x \longrightarrow \pm\infty} [f(x) - \pm\, x] = \lim\limits_{x \longrightarrow \pm\infty} [\sqrt{x^2 - x} - \sqrt{x^2}] =$ [multiply and divide by the conjugate:]
  $= \lim\limits_{x \longrightarrow \pm\infty} [\sqrt{x^2 - x} - \sqrt{x^2}][\sqrt{x^2 - x} + \sqrt{x^2}]/[\sqrt{x^2 - x} + \sqrt{x^2}] =$
  $= \lim\limits_{x \longrightarrow \pm\infty} \dfrac{x^2 - x - x^2}{\sqrt{x^2 - x} + \sqrt{x^2}} = - \lim\limits_{x \longrightarrow \pm\infty} \dfrac{x}{\sqrt{x^2 - x} + \sqrt{x^2}} =$
  [dividing both numerator and denominator by $x = \sqrt{x^2}$]
  $= - \lim\limits_{x \longrightarrow \pm\infty} \dfrac{\pm 1}{\sqrt{1 - 1/x} + 1} = \mp \frac{1}{2};$
- Hence, f has $y = x - \frac{1}{2}$ as oblique asymptote at ∞, and $y = x + \frac{1}{2}$ at $-\infty$.

Since $f(1) = 0$, $f(x) \geq 0$, f is continuous and that $y = x - \frac{1}{2}$ is the oblique asymptote at ∞, the image of f is $[0, \infty)$.

(b) $f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x}}$, hence f is increasing in $[1, \infty)$ and decreasing in $(-\infty, 0]$. Moreover, $f''(x) = \left(\dfrac{2x - 1}{2\sqrt{x^2 - x}} \right)' = \dfrac{4\sqrt{x^2 - x} - (2x-1)^2/\sqrt{x^2 - x}}{4(x^2 - x)} < 0$, for all x in $(-\infty, 0) \cup (1, \infty)$, thus f is concave both in $(-\infty, 0]$ and in $[1, \infty)$ (f is continuous, thus we can consider the closed intervals).

Note that $4\sqrt{x^2 - x} < (2x-1)^2/\sqrt{x^2 - x} \iff 4(x^2 - x) < (2x-1)^2 \iff 0 < 1$.



2

Given the parameters $a, b \in \mathbb{R}$, consider the function $f(x) = \begin{cases} 1 + ax^2, & \text{if } x < -1; \\ abx & \text{if } x \geq -1. \end{cases}$

(a) (15 points) For what values of the parameters a and b the function f is continuous and differentiable? Justify your answer.

(b) (15 points) For the values of the parameters $a = -\frac{1}{2}$ and $b = 1$, find the global and local extrema of the function f in the interval $[-2, 0]$.

Solution:

(a) Let us study the continuity and differentiability of the function f at the point $x = -1$, since at other points it is clearly continuous.

Given that $\lim_{x \longrightarrow -1^-} f(x) = 1 + a$, $f(-1) = \lim_{x \longrightarrow -1^+} f(x) = -ab$, we obtain that the function is continuous at $x = -1$ if and only if $1 + a = -ab$.

On the other hand, assuming that f is continuous at -1, it shall be differentiable if and only if $-2a = f'_-(-1) = f'_+(-1) = ab$.

Hence, f is continuous and differentiable at $x = -1$ if and only if $1 + a = -ab, -2a = ab$. We have the following cases.

- if $a = 0$, then the first equation above is not fulfilled;
- if $a \neq 0$, then from the second equation above, $b = -2$ and then, from the first one, $a = 1$.

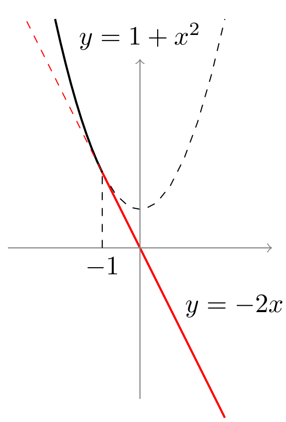
Hence, f is continuous and differentiable at $x = -1$ if and only if $a = 1$ and $b = -2$.

(b) The parameters $a = -\frac{1}{2}$ and $b = 1$ fulfill the continuity condition $1 + a = -ab$ found in the item above. Hence, f is continuous. By the Weierstrass Theorem, f attains global extrema in $[-2, 0]$.

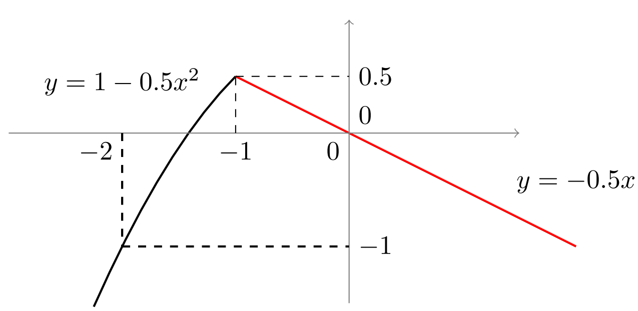
On the other hand, the differentiability condition $-2a = ab$ is not fulfilled, thus f is not differentiable at $x = -1$, thus -1 is a critical point of f. To explore more critical points, we let $f'(x) = -x$, if $x < -1$ and $f'(x) = -\frac{1}{2}$, if $x > -1$. Hence, f has not more critical points.

The candidates for global extrema of f in $[-2, 0]$ are thus -2, -1 and 0, with $f(-2) = -1$, $f(-1) = \frac{1}{2}$ and $f(0) = 0$. We obtain that $x = -2$ is the global minimum, and $x = -1$ is the global maximum of f in $[-2, 0]$. Regarding the point $x = 0$, it is a local minimum of f in $[-2, 0]$, since that f is decreasing in the interval $[-1, \infty)$.

Case (a)



Case (b)



3

Answer the following questions.

(a) (15 points) Calculate the Taylor polynomial of order two of the function $f(x) = (1+x)^{10}$ at the point $x = 0$. With the Taylor polynomial, calculate an approximated value of the number $(1.02)^{10}$.

(b) (15 points) Prove that the equation $ye^{-x} - y^3 - 3x = 0$ defines y as an implicit function of x, $y = f(x)$, around the point $(0, 1)$. Calculate the derivative of the implicit function $y = f(x)$ at the point 0, $f'(0)$.

Solution:

(a) The Taylor polynomial of second order of f at $x = 0$, is $P(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2$. Computing the coefficients we obtain $P(x) = 1 + 10x + 45x^2$. The approximated value of $(1.02)^{10}$ is

$$(1.02)^{10} = f(0.02) \approx P(0.02) = 1 + 10 \times 0.02 + 45 \times (0.02)^2 = 1 + 0.2 + 0.0180 = 1.2180.$$

(b) Note that $(x, y) = (0, 1)$ satisfies $ye^{-x} - y^3 - 3x = 0$. Also, if we let $F(x, y) = ye^{-x} - y^3 - 3x$, then $\frac{\partial F}{\partial y}(x, y) = e^{-x} - 3y^2$, and $\frac{\partial F}{\partial y}(0, 1) = -2 \neq 0$, hence $F(x, y) = 0$ defines $y = f(x)$ around $x = 0$, with $f(0) = 1$. Deriving $F = 0$ with respect to x, $y'e^{-x} - ye^{-x} - 3y'y^2 - 3 = 0$, and evaluating this expression at $x = 0$ and $y = 1$, we obtain $y' - 1 - 3y' - 3 = 0$. Solving, $y' = f'(0) = -2$.

4

Given the cost function $C(x) = 4000 - 40x + 0.02x^2$ and the inverse demand function $p(x) = 50 - 0.01x$, where x is the number of units of a good produced by a monopolistic firm, answer the following questions.

(a) (15 points) Calculate the unitary price, p^*, and the amount of good produced, x^*, which maximizes the profits. Justify your findings.

(b) (15 points) Calculate the unitary price p^{**}, and the amount of good produced, x^{**}, which minimizes the average cost. Justify your findings.

Solution:

(a) The profit function is $\pi(x) = (50 - 0.01x)x - (4000 - 40x + 0.02x^2) = -0.03x^2 + 90x - 4000$. The first and second order derivatives are $\pi'(x) = -0.06x + 90$ and $\pi''(x) = -0.06 < 0$, for all x. Hence π has a unique critical point at $x^* = \dfrac{90}{0.06} = 1500$. Since π is concave, this critical point is a unique maximizer of the firm's profits. Finally, $p^* = p(1500) = 50 - 0.01 \times 1500 = 35$.

(b) The average cost function is $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{4000}{x} - 40 + 0.02x$. The first and second order derivatives are $\overline{C}'(x) = -\dfrac{4000}{x^2} + 0.02$, and $\overline{C}''(x) = \dfrac{8000}{x^3} > 0$, for all $x > 0$. The only critical point of $\overline{C}$ in the region $x > 0$ is $x^{**} = \sqrt{5} \times 200$. Since $\overline{C}$ is convex, the critical point is a unique minimizer of the average cost function. Finally, $p^{**} = p(\sqrt{5} \times 200) = 50 - 0.01 \times \sqrt{5} \times 200 = 50 - 2\sqrt{5}$.