1

Consider the matrix

$$A = \begin{pmatrix} 3a & 0 & 0 \\ a & 2a & -1 \\ a^2 & -a^2 & 2a \end{pmatrix}$$

where a is a real valued parameter.

(a) Find the eigenvalues and eigenvectors of A depending on the values of the parameter a.

(b) For which values of a is the matrix A diagonalizable? For the values of a for which the matrix A diagonalizable, find matrices D and P such that $A = PDP^{-1}$.

(c) Plugging $a = 1$ into the matrix A, compute the general solution of the following system of difference equations and study its stability.

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

You may leave A^t as the product of three matrices.

Solution:

1. The eigenvalues of A are $a, 3a, 3a$.
   For $a \neq 0$ the eigenvectors are

$$\begin{aligned} S(a) &= \, < (0,1,a) > \\ S(3a) &= \, < (1,0,a)\,, (1,1,0) > \end{aligned}$$

   For $a = 0$ the matrix A is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

   The eigenvalues are $0, 0, 0$ and the eigenvectors are $S(0) =< (1,0,0), (0,1,0) >$.

2. For $a \neq 0$, the matrix A is diagonalizable, $A = PDP^{-1}$ with

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 3a & 0 \\ 0 & 0 & 3a \end{pmatrix} \quad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & a & 0 \end{pmatrix}$$

   For $a = 0$, the matrix is not diagonalizable.

3. The matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

   is obtained by plugging in the value $a = 1$ in the matrix A above. Thus

$$I - A = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

   and we solve the system of equations

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

   The solution is $x = -1$, $y = 0$, $z = 0$.

We have shown that

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}^t = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^t & 0 \\ 0 & 0 & 3^t \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

Thus, the general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}^t \left(\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

with C_1, C_2, C_3 arbitrary real numbers. Since the eigenvalues are $1, 3, 3$, the system is unstable.

2

Consider the equation

$$x_{t+2} - x_{t+1} - 6x_t = 5 + 6t.$$

(a) Find the general solution.

(b) Find the solution satisfying $x_0 = 0$, $x_1 = 0$.

Solution:

1. The characteristic polynomial is $r^2 - r - 6$ whose roots are $r_1 = 2$, $r_2 = -3$. Thus, the general solution of

$$x_{t+2} - x_{t+1} - 6x_t = 5 + 6t$$

is $C_1 2^t + C_2(-3)^t$ with C_1, C_2 arbitrary real numbers. Now we look for a solution of the form $y_t = at + b$ of the equation $x_{t+2} - x_{t+1} - 6x_t = 5 + 6t$. We see that there is a solution with $a = b = -1$. Hence, the general solution is

$$x_t = C_1 2^t + C_2(-3)^t - t - 1.$$

2. Plugging in the values $t = 0$ and $t = 1$ in the general solution we have that

$$\begin{aligned} 0 &= c_1 + c_2 - 1 \\ 0 &= 2c_1 - 3c_2 - 2 \end{aligned}$$

and solving this system of equations we obtain $c_1 = 1$, $c_2 = 0$ Hence, the solution is $x_t = 2^t - t - 1$.

3

(a) Find the solution of

$$t^2 y'(t) + y(t) = 1.$$

(b) Find the solution of

$$y'(t) = \frac{t^2}{y(t)}$$

satisfying $y(0) = 1$.

Solution:

1. The equation is linear. The general solution is

$$y(t) = C_1 e^{\frac{1}{t}} + 1.$$

2. The equation is separable. The general solution satisfies

$$y^2(t) = \frac{2}{3}t^3 + C.$$

The solution is $y(t) = \sqrt{\frac{2}{3}t^3 + 1}$.

4

(a) Consider the differential equation

$$y' = y^3 - y.$$

Find and classify its stationary points.

(b) Let $y(t)$ be the solution of initial value problem

$$y' = y^3 - y, \quad y(0) = \frac{1}{2}$$

. Compute $\lim_{t\to-\infty} y(t)$ and $\lim_{t\to\infty} y(t)$. Can you determine if $y(t)$ is increasing or decreasing?

Solution:

1. Note that $f(y) = y^3 - y = y(y^2 - 1) = y(y-1)(y+1)$. Then the stationary points are -1, 0, 1. Note that $f < 0$ for $y < -1$, $f > 0$ for $-1 < y < 0$, $f < 0$ for $0 < y < 1$ and $f > 0$ for $y > 1$. Thus, we conclude that -1 and 1 are unstable and 0 is locally asymptotically stable.

2. $y(0) = 1/2 \in (0, 1)$ is in the region of stability of the differential equation, thus we see $\lim_{t\to-\infty} y(t) = 1$, $\lim_{t\to\infty} y(t) = 0$; moreover, $y(t)$ is decreasing, since $f(1/2) < 0$.

5

Consider the following linear system of differential equations

$$\begin{cases} \dot{x} &= x + by \\ \dot{y} &= bx + y \end{cases}$$

with $b \in \mathbb{R}$. Compute the equilibrium points and classify its stability depending on the values of b.

Solution:

The equilibrium point is $(0, 0)$. The matrix associated to the linear system of differential equations is

$$A = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = 1 - b$ and $\lambda_2 = 1 + b$.
If $-1 < b < 1$, then $\lambda_1 > 0$ and $\lambda_2 > 0$ and $(0, 0)$ is an unstable node.
If $b = -1$ or $b = 1$, then the eigenvalues are 0 and 2, so $(0, 0)$ is an unstable node.
If $b < -1$, then $\lambda_1 > 0$ and $\lambda_2 < 0$, or if $b > 1$, then $\lambda_1 < 0$ and $\lambda_2 > 0$, thus $(0, 0)$ is a saddle point.