UC3M
Advanced Mathematics for Economics
Final Exam, 19 June 2017 SOLVED

1

Find the solution of the following difference equation with initial conditions

$$x_{t+2} + x_{t+1} - 6x_t = 4t, \quad x_0 = -\frac{3}{4}, \quad x_1 = -\frac{3}{4}.$$

Solution:

The characteristic equation is $r^2 + r - 6 = 0$, whose roots are 2 and -3. Thus the general solution of the associated homogeneous equation is

$$x_t^h = C_1 2^t + C_2(-3)^t$$

Since 1 is not a root of characteristic equation we look for a particular solution of the form $x_t^p = At + B$. By substituting x_t^p into the equation we find that $A = -1$ and $B = -\frac{3}{4}$. Thus, the general solution is

$$x_t^g = C_1 2^t + C_2(-3)^t - t - \frac{3}{4}.$$

Plugging the values $t = 0$ and $t = 1$ we get the following system of linear equations

$$\begin{aligned} -\frac{3}{4} &= C_1 + C_2 - \frac{3}{4} \\ -\frac{3}{4} &= 2C_1 - 3C_2 - 1 - \frac{3}{4} \end{aligned}$$

whose solution is $C_1 = \frac{1}{5}$, $C_2 = -\frac{1}{5}$. Hence the solution is

$$x_t = \frac{1}{5}2^t - \frac{1}{5}(-3)^t - t - \frac{3}{4}.$$

2

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & a \\ 0 & 0 & b \end{pmatrix}$$

where $a, b \in \mathbb{R}$.

(a) (10 points) For what values of the parameters a and b is the matrix A diagonalizable?

(b) (10 points) For the values of the parameters a and b for which the matrix A is diagonalizable, write its diagonal form and the diagonalization matrix P.

Solution:

(a) The eigenvalues of A are $\frac{1}{2}$ and b. When $b = \frac{1}{2}$, the eigenvalue $\frac{1}{2}$ has multiplicity 3. Since the rank of $A - \frac{1}{2}I$ is 1, the matrix is not diagonalizable in this case. When $b \neq \frac{1}{2}$ the eigenvalue $\frac{1}{2}$ has multiplicity 2. The matriz $A - \frac{1}{2}I$ is

$$A - \frac{1}{2}I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 0 & 0 & b - \frac{1}{2} \end{pmatrix},$$

which has rank 1, thus A is diagonalizable.

(b) Let $b \neq \frac{1}{2}$. We have $S(\frac{1}{2}) = \langle (1,0,0), (0,1,0) \rangle$ and $S(b) = \left\langle \left(-\frac{1}{\frac{1}{2}-b}, -\frac{a}{\frac{1}{2}-b}, 1 \right) \right\rangle$. The diagonal matrix is

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & b \end{pmatrix}$$

and the diagonalization matrix is

$$P = \begin{pmatrix} 1 & 0 & -\frac{1}{\frac{1}{2}-b} \\ 0 & 1 & -\frac{a}{\frac{1}{2}-b} \\ 0 & 0 & 1 \end{pmatrix}$$

3

Consider the following system of difference equations.

$$\begin{aligned} x_{t+1} &= \frac{x_t}{2} + z_t + 2 \\ y_{t+1} &= \frac{y_t}{2} + a z_t + 1 \\ z_{t+1} &= b z_t \end{aligned}$$

with $a, b \in \mathbb{R}$ and $b \neq 1$, $b \neq \frac{1}{2}$.

(a) (5 points) Find the equilibrium point of the system of difference equations.

(b) (10 points) Compute the general solution of the above system of difference equations.

(c) (5 points) For what values of the parameters a and b is the above system globally asymptotically stable? For those values of a and b for which the above system of difference equations is globally asymptotically stable compute the limit of the trajectories.

Solution:

(a) The equilibrium point is

$$x^0 = (4, 2, 0)$$

1. The matrix associated to the system of difference equations is the matrix A of the previous exercise, with $b \neq \frac{1}{2}$, thus A is diagonalizable. The general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = c_1 \left(\frac{1}{2}\right)^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \left(\frac{1}{2}\right)^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 b^t \begin{pmatrix} -\frac{1}{\frac{1}{2}-b} \\ -\frac{a}{\frac{1}{2}-b} \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

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2. The system of difference equations is globally asymptotically stable for the values $|b| < 1$. For those values of b we have

$$\lim_{t \to \infty} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}.$$

4

Find an integrating factor of the differential equation

$$(xt^2 - x^3)dt + (x^2t - t^3)dx = 0$$

and give the general solution.

Solution:

Let $P = xt^2 - x^3$ and let $Q = x^2t - t^3$. Note that $\frac{\partial P}{\partial x} = t^2 - 3x^2$ and that $\frac{\partial Q}{\partial t} = x^2 - 3t^2$ hence the differential equation is not exact. The ratio

$$\frac{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial t}}{Q} = \frac{(t^2 - 3x^2) - (x^2 - 3t^2)}{x^2t - t^3} = \frac{4(t^2 - x^2)}{t(x^2 - t^2)} = -\frac{4}{t}$$

is independent of x. Thus, an integrating factor is $\mu(t) = \exp \int -\frac{4}{t} dt = \frac{1}{t^4}$. Multipliying the equation by $\frac{1}{t^4}$ it becomes exact. Let

$$V(t,x) = \int \left(\frac{xt^2 - x^3}{t^4} \right) dt = x \int t^{-2} dt - x^3 \int t^{-4} dt = -xt^{-1} + \frac{1}{3}x^3t^{-3} + f(x).$$

By imposing $\frac{\partial V}{\partial x} = \frac{x^2t - t^3}{t^4}$, we find

$$-t^{-1} + x^2t^{-3} + f'(x) = x^2t^{-3} - t^{-1},$$

hence $f'(x) = 0$ and then we choose $f(x) = 0$. The general solution is given by

$$-xt^{-1} + \frac{1}{3}x^3t^{-3} = C, \qquad C \text{ constant.}$$

5

(a) (5 points) Find the general solution of the following ODE

$$x'' - x' - 6x = 8 - 2t - 6t^2$$

(b) (5 points) Find the solution $x(t)$ of the the above ODE that satisfies the following initial conditions

$$x(0) = 5, \quad \dot{x}(0) = -2.$$

Solution:

(a) The characteristic equation is $r^2 - r - 6 = 0$ whose roots are -2 and 3. Hence, the general solution of the associated homogeneous equation is

$$x^h(t) = c_1 e^{-2t} + c_2 e^{3t}$$

We look now for a particular solution of the form

$$y(t) = At^2 + Bt + C$$

Thus,

$$\begin{aligned} y'(t) &= 2At + B \\ y''(t) &= 2A \\ y'' - y' - 6y &= 2A - 2At - B - 6At^2 - 6Bt - 6C \end{aligned}$$

and we obtain $2A - B - 6C = 8$, $-2A - 6B = -2$ and $-6A = -6$. Solving, we find $A = 1$, $B = 0$ and $C = -1$. Hence, the general solution is

$$x^g(t) = c_1 e^{-2t} + c_2 e^{3t} + t^2 - 1$$

(b) Note that

$$\dot{x}^g(t) = -2c_1 e^{-2t} + 3c_2 e^{3t} + 2t.$$

Plugging the values $x^g(0) = 5$ and $\dot{x}^g(0) = -2$ into the general solution, we get the system

$$\begin{cases} c_1 + c_2 - 1 & = 5 \\ -2c_1 + 3c_2 & = -2 \end{cases}$$

Solving, we find $c_1 = 4$ and $c_2 = 2$. Hence, the solution is

$$x(t) = 4e^{-2t} + 2e^{3t} + t^2 - 1$$

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6

Consider the autonomous ordinary differential equation

$$x' = F(x),$$

where $F(x) = (x+3)(2-x)(x-5)$.

(a) (10 points) Determine and classify its stationary points.

(b) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = -5$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(c) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = 0$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(d) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = 5$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(e) (5 points) Let $x(t) = 3 - t^2$. Discuss whether this function could be solution of the following initial value problem

$$x' = F(x), \quad x(0) = 3$$

Solution:

(a) The stationary points are -3 (l.a.s.), 2 (unstable) and 5 (l.a.s.).

(b)
1. $x(t)$ is increasing.
2. $\lim_{t\to\infty} x(t) = -3$ and $\lim_{t\to-\infty} x(t) = -\infty$.
3.

(c)
1. $x(t)$ is decreasing.
2. $\lim_{t\to\infty} x(t) = -3$ and $\lim_{t\to-\infty} x(t) = 2$.
3.

(d)
1. $x(t) = 5$ for all values of t.
2. $\lim_{t\to\infty} x(t) = \lim_{t\to-\infty} x(t) = 5$.
3.

(e) $x(t) = 3 - t^2$ cannot be a solution, since any solution with $2 < x(0) < 5$ must be increasing, but $3 - t^2$ is decreasing.