UC3M
Advanced Mathematics for Economics
Final Exam, 9 January 2017 SOLVED

1

Find the solution of the following difference equation with initial conditions

$$x_{t+2} + 3x_{t+1} - 4x_t = 15, \quad x_0 = 1, \quad x_1 = -1.$$

Solution:

The characteristic equation is $r^2 + 3r - 4 = 0$, whose roots are 1 and -4. Thus the general solution of the associated homogeneous equation is

$$x_t^h = C_1 + C_2(-4)^t$$

Since 1 is a root of characteristic equation we look for a particular solution of the form $x_t^p = At$. Substituting this form of x_t^p into the equation we find that $A = 3$. Thus, the general solution is

$$x_t^g = C_1 + C_2(-4)^t + 3t$$

Plugging the values $t = 0$ and $t = 1$ we get the following system of linear equations

$$\begin{aligned} 1 &= C_1 + C_2 \\ -1 &= C_1 + -4C_2 + 3 \end{aligned}$$

whose solution is $C_1 = 0$, $C_2 = 1$. Hence the solution is

$$x_t = (-4)^t + 3t$$

2

Consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ a^2 - \frac{a}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

where $a \in \mathbb{R}$.

(a) (10 points) For what values of the parameter a is the matrix A diagonalizable?

(b) (10 points) For the values of parameter a for which the matrix A diagonalizable, write its diagonal form.

Solution:

$$P = \begin{pmatrix} 1 & 0 & o \\ 0 & 1 & a \\ a & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

The matrix is diagonalizable for every value of $a \in \mathbb{R}$.

3

Consider the following system of difference equations.

$$\begin{aligned} x_{t+1} &= ax_t + 1 \\ y_{t+1} &= \frac{y_t}{2} + 1 \\ z_{t+1} &= \left(a^2 - \frac{a}{2}\right) x_t + \frac{z_t}{2} \end{aligned}$$

with $a \in \mathbb{R}$ and $a \neq 1$.

(a) (5 points) Find the equilibrium point of the system of difference equations.

(b) (10 points) Compute the general solution of the above system of difference equations.

(c) (5 points) For what values of the parameter a is the above system globally asymptotically stable? For those values of a for which the above system of difference equations is globally asymptotically stable compute the limit of the trajectories.

Solution:

1. The equilibrium point is

$$x^0 = (\frac{1}{1-a}, 2, \frac{2a^2 - a}{a-1})$$

2. The matrix associated to the system of difference equations is the matrix A of the previous exercise. Thus, the general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = c_1 a^t \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} + \frac{c_2}{2^t} \begin{pmatrix} 0 \\ 1 \\ o \end{pmatrix} + \frac{c_3}{2^t} \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{1-a} \\ 2 \\ \frac{2a^2-a}{a-1} \end{pmatrix}$$

.

3. The system of difference equations is globally asymptotically stable for the values $|a| < 1$. For those values of a we have

$$\lim_{t\to\infty} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} \frac{1}{1-a} \\ 2 \\ \frac{2a^2-a}{a-1} \end{pmatrix}.$$

4

Compute the solution of the following initial value problem.

$$\dot{x} - x = -t, \quad x(0) = x_0.$$

Solution:

Multiplying the equation by e^{-t} it becomes

$$\frac{d}{dt}\left(e^{-t}x\right) = -te^{-t}$$

So,

$$e^{-t}x = te^{-t} + e^{-t} + A$$

Plugging in $t = 0$, $x(0) = x_0$ we obtain $x_0 = 1 + A$. Hence,

$$x(t) = t + 1 + (x_0 - 1)e^t$$

5

(a) (5 points) Find the general solution of the following ODE

$$x'' - x' - 6x = 16te^{-t}$$

(b) (5 points) Find the solution $x(t)$ of the the above ODE that satisfies the following initial conditions

$$x(0) = 4, \qquad \lim_{t \to +\infty} x(t) = 0$$

Solution:

1. The characteristic equation is $r^2 - r - 6 = 0$ whose roots are -2 and 4. Hence, the general solution of the associated homogeneous equation is

$$x^h(t) = c_1 e^{-2t} + c_2 e^{3t}$$

We look now for a particular solution of the form

$$y(t) = (A + Bt)e^{-t}$$

Thus,

$$\begin{aligned} y'(t) &= (A - B + Bt)e^{-t} \\ y''(t) &= (A - 2B + Bt)e^{-t} \\ y'' - y' - 6y &= (4A + 3B + 4Bt)e^{-t} \end{aligned}$$

and we obtain $A = 3$, $B = -4$. Hence, the general solution is

$$x^g(t) = c_1 e^{-2t} + c_2 e^{3t} + (3 - 4t)e^{-t}$$

2. Since, $\lim_{t \to +\infty} x(t) = 0$ we must have $c_2 = 0$. And since $x(0) = 4$ we have that $4 = c_1 + 3$ and we obtain that $c_1 = 1$. Hence, the solution is

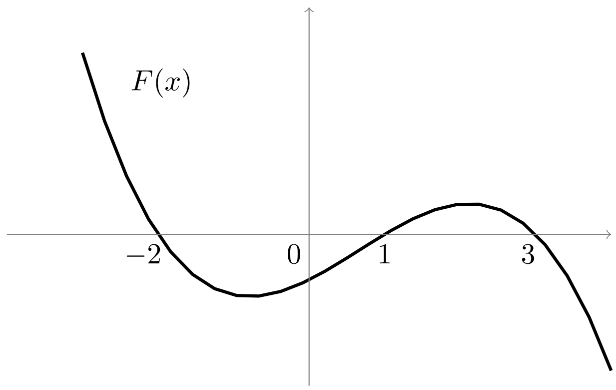
$$x(t) = e^{-2t} + (3 - 4t)e^{-t}$$

6

Consider the autonomous ordinary differential equation

$$x' = F(x),$$

where the graph of the function F is showed in the figure below.



(a) (10 points) Determine and classify its stationary points.

(b) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = -3$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(c) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = 0$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(d) (5 points) Let $x(t)$ be the solution of the following initial value problem

$$x' = F(x), \quad x(0) = 1$$

1. Is $x(t)$ increasing or decreasing?
2. Compute $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.
3. Sketch the graph of $x(t)$.

(e) (5 points) Let $x(t) = 4e^t$. Discuss whether this function could be solution of the following initial value problem

$$x' = F(x), \quad x(0) = 4$$

Solution:

(a) The stationary points are -2 (lae), 1 (unstable) and 3 (lae).

(b)
1. $x(t)$ is increasing.
2. $\lim_{t\to\infty} x(t) = -2$ and $\lim_{t\to-\infty} x(t) = -\infty$.
3.

(c)
1. $x(t)$ is decreasing.
2. $\lim_{t\to\infty} x(t) = -2$ and $\lim_{t\to-\infty} x(t) = 1$.
3.

(d)
1. $x(t) = 1$ for all values of t.
2. $\lim_{t\to\infty} x(t) = \lim_{t\to-\infty} x(t) = 1$.
3.

(e) $x(t) = 4e^t$ cannot be a solution, since any solution with $x(0) > 3$ must be decreasing, but $4e^t$ is increasing.