$\boxed{1}$

Consider the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & a & 1 \end{pmatrix},$$

where $a \in \mathbb{R}$.

(a) (5 points) Study whether A is diagonalizable or not depending on the values of the parameter a.

(b) (10 points) For the case or cases where A is diagonalizable, find its diagonal form and the matrix P that diagonalize A.

(c) (10 points) For the case or cases where A is diagonalizable, solve the system of difference equations

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Solution:

(a) The eigenvalues are 2 and 1, with 1 being of multiplicity 2. The matrix $A - I$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & a & 0 \end{pmatrix}.$$

It has rank 1 iff $a = 0$. Hence the matrix is diagonalizable iff $a = 0$.

(b) Let $a = 0$, so that the matrix A becomes

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We easily find

$$S(2) = \langle (1, 1, -1) \rangle, \qquad S(1) = \langle (0, 1, 0), (0, 0, 1) \rangle.$$

The matrix P is formed using the above eigenvectors as columns and then P^{-1} is easily found

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(c) We have

$$A^t = PD^tP^{-1} = \begin{pmatrix} 2^t & 0 & 0 \\ 2^t - 1 & 1 & 0 \\ 1 - 2^t & 0 & 1 \end{pmatrix},$$

where D is the diagonal matrix associated to A. Hence the solution is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 2^t & 0 & 0 \\ 2^t - 1 & 1 & 0 \\ 1 - 2^t & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2^t \\ 2^t \\ 3 - 2^t \end{pmatrix}.$$

$\boxed{2}$

Consider the following model that links costumers' bank deposits with the interest rate offered by the bank. The amount of deposits at time t is denoted D_t and the interest rate is R_t. Deposits become more attractive for costumers if R_t increases. We postulate that the evolution of deposits is given by

$$D_{t+1} = D_t + a(R_t - R), \qquad a > 0,$$

where $R > 0$ is a threshold interest rate above (below) which people hold more (less) deposits. The deposit interest rate offered by the bank is decreasing with the amount of deposits. We postulate that the evolution of the interest rate is

$$R_{t+1} = S - bD_t, \qquad b > 0,$$

where $S > R$ is the maximum deposit interest rate that the bank is able to offer.

(a) (5 points) Find a difference equation of order 2 for bank deposits.

(b) (10 points) Solve the equation in (a) for the case $ab = \frac{1}{2}$ and study $\lim_{t\to\infty} D_t$.

(c) (10 points) Solve the equation in (a) for the case $ab = \frac{1}{4}$ and study $\lim_{t\to\infty} D_t$.

Solution:

(a) $D_{t+2} - D_{t+1} + abD_t = a(S - R)$.

(b) The particular solution is the constant $D^* = (S - R)/b$. The characteristic equation has solutions $r_{1,2} = \frac{1\pm\sqrt{1-4ab}}{2}$. When $ab = 1/2$, both roots are complex conjugates, $r_{1,2} = \frac{1\pm i}{2}$, with $\rho = \frac{\sqrt{2}}{2}$ and $\theta = \pi/4$, thus

$$D_t = 2^{-\frac{t}{2}} \left(A \cos \frac{\pi}{4} t + B \sin \frac{\pi}{4} t \right) + \frac{S - R}{b}.$$

$\lim_{t\to\infty} D_t = \frac{S-R}{b}$.

(c) When $ab = 1/4$, the root is double, $r_{1,2} = 1/2$ and hence

$$D_t = A2^{-t} + Bt2^{-t} + \frac{S - R}{b}.$$

$\lim_{t\to\infty} D_t = \frac{S-R}{b}$.

$\boxed{3}$

(a) (8 points) Solve

$$x' = \frac{2tx}{(t^2+2)(1+x)}, \quad x(0) = 1.$$

(b) (8 points) Find the general solution of $(x + t^4)dt - tdx = 0$.

(c) (8 points) Find the general solution of $x'' - x' + \frac{5}{4}x = 1 + 5t + 13e^t$.

Solution:

(a) Separable

$$\int \frac{1+x}{x} dx = \int \frac{2t}{t^2+2} dt,$$

hence

$$x + \ln x = \ln(t^2+2) + C.$$

Now, $x(0) = 1$ implies $C = 1 - \ln 2$.

(b) It is not exact, but t^{-2} is an integrating factor. The solution is $-\frac{x}{t} + \frac{t^3}{3} = C$, or $x = \frac{t^4}{3} - Ct$, with C constant.

(c) Linear, constant coefficients, second order. The characteristic equation is $r^2 - r + \frac{5}{4} = 0$, with solutions $\frac{1}{2} \pm i$. The general solution of the complete equation is $e^{t/2}(C_1 \sin t + C_2 \cos t) + 4 + 4t + \frac{52}{5}e^t$.

$\boxed{4}$

A firm produces a single good that sells in a market. Let $x(t)$ be the sales at time t. Without advertisement, sales diminish at rate $\delta > 0$. However, by applying an advertisement effort of $a > 0$ monetary units, the marginal increment of sales is

$$a\left(\frac{M - x(t)}{M}\right).$$

Here, $M > 0$ is an upper saturation level that sells cannot exceed, as the number of possible costumers is limited. Thus, the evolution of total sales is given by

$$x'(t) = -\delta x(t) + a\left(\frac{M - x(t)}{M}\right), \tag{1}$$

and the sales at time $t = 0$ are $x(0) = x_0$, with $0 \leq x_0 < M$.

(a) (10 points) Find the solution of the linear equation (1). Show that (1) has a unique equilibrium point, x^0, and find it. Is it globally asymptotically stable?

(b) (6 points) Suppose that at time $t = 0$ the initial sales of the firm satisfy $0 < x_0 < \dfrac{M}{2}$. Find the advertisement effort a, such that the sales in the long run are twice x_0, that is, $x^0 = 2x_0$.

(c) (10 points) Assume now that the advertisement effort is no longer constant, but proportional to sales, $a(x) = px$, with $\underline{p > \delta}$. The evolution of sales is thus

$$\dot{x}(t) = -\delta x(t) + px(t)\left(\frac{M - x(t)}{M}\right).$$

This equation in no longer linear. Find the nonnegative equilibrium points (if any) and study their stability. Justify your answer by sketching the phase diagram.

Solution:

(a) It is $x(t) = x^0 + (x_0 - x^0)e^{-kt}$, where $x^0 = \frac{aM}{a+\delta M}$ is the equilibrium point and $k = \delta + \frac{a}{M}$. Yes, it is g.a.s., as $k > 0$.

(b) We must solve the equation

$$\frac{aM}{a + \delta M} = 2x_0$$

to find

$$a = \frac{2\delta M x_0}{M - 2x_0}.$$

(c) The equilibrium points are solutions of $x\left(-\delta + p\left(1 - \frac{x}{M}\right)\right) = 0$, thus we get $x_1^0 = 0$ and $x_2^0 = M\left(1 - \frac{\delta}{p}\right) > 0$, since $p > \delta$. Since the r.h.s. of the ODE is a concave parabola, it is clear that x_1^0 is unstable and that x_2^0 is l.a.s., see the figure below.

Graph of f, x'=f(x)