1

Answer the following questions.

(a) (5 points) Find the general solution of the second order difference equation

$$x_{t+2} - 2x_{t+1} + 2x_t = 0.$$

(b) (5 points) Find the general solution of the second order difference equation

$$x_{t+2} - 2x_{t+1} + 2x_t = t.$$

Solution:

(a) The characteristic equation is $r^2 - 2r + 2 = 0$. The solutions are complex, $1 \pm i$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i\sqrt{4}}{2} = 1 \pm i.$$

The module of $1+i$ is $\sqrt{2}$ and the argument $\theta = \arctan(1/1) = \arctan 1 = \pi/4$.

Thus the solution is

$$\rho^t(C_1 \cos\theta t + C_2 \sin\theta t) = (\sqrt{2})^t(C_1 \cos(t\pi/4) + C_2 \sin(t\pi/4)).$$

(b) A particular solution has the form $At + B$. This is a solution iff

$$A(t+2) + B - 2A(t+1) - 2B + 2At + 2B = t.$$

This is possible iff $A - 2A + 2A = 1$ and $2A - 2A + B - 2B + 2B = 0$. Thus, $A = 1$ and $B = 0$.

We have already solved the homogenous equation, thus the general solution is

$$(\sqrt{2})^t(C_1 \cos(t\pi/4) + C_2 \sin(t\pi/4)) + t.$$

2

Consider the matrix

$$A = \begin{pmatrix} 2 & \frac{9}{a} & 3 \\ a & 2 & a \\ 0 & 0 & -1 \end{pmatrix},$$

where $a \neq 0$.

(a) (5 points) For what values of the parameter a is the matrix A diagonalizable?

(b) (5 points) Calculate the eigenvalues and eigenvectors of A. Write the diagonal form of A and the matrix P associated when A is diagonalizable.

Solution:

(a) Eigenvalues: -1 double and 5 simple.

This is because the characteristic polynomial is $|A - \lambda I| = (-1-\lambda)((2-\lambda)^2 - 9) = -(\lambda+1)(\lambda+1)(\lambda-5)$.

A will be diagonalizable iff the rank of $A + I$ is 1. The matrix $A + I$ is

$$A + I = \begin{pmatrix} 3 & \frac{9}{a} & 3 \\ a & 3 & a \\ 0 & 0 & 0 \end{pmatrix}.$$

The two non-null rows are proportional, thus the rank is 1. In consequence, A is diagonalizable for all $a \neq 0$.

(b) • $S(-1)$ is generated by $(1, 0, -1)$ and $(-3, a, 0)$.

To prove this, we solve the system $(A + I)\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} 3 & \frac{9}{a} & 3 \\ a & 3 & a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It reduces to the single equation $ax + 3y + az = 0$. The solutions are as described above.

• $S(5)$ is generated by $(3, a, 0)$.

To prove this, we solve the system $(A - 5I)\mathbf{x} = \mathbf{0}$:

$$\begin{pmatrix} -3 & \frac{9}{a} & 3 \\ a & -3 & a \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get $z = 0$ and $ax - 3y + az = 0$ as independent equations. The solutions are as described above.

Thus, we can choose

$$D = \operatorname{diag}(-1,\ -1,\ 5)$$

and then

$$P = \begin{pmatrix} 1 & -3 & 3 \\ 0 & a & a \\ -1 & 0 & 0 \end{pmatrix}.$$

3

Consider the following linear system of difference equations,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 2 & \frac{9}{a} & 3 \\ a & 2 & a \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \\ 0 \end{pmatrix},$$

with $a \neq 0$. Note that the matrix of the system is the matrix A of Problem 2.

(a) (5 points) Find the equilibrium point.

(b) (5 points) For the values of a for which the matrix is diagonalizable find the general solution.

Solution:

(a) The equilibrium point is the solution of the system

$$\begin{cases} x &= 2x + \frac{9}{a}y + 3z + 8 \\ y &= ax + 2y + az \\ z &= -z \end{cases}$$

From the third equation $z = 0$ and the two first equations become

$$\begin{cases} x + \frac{9}{a}y &= -8 \\ ax + y &= 0 \end{cases}$$

Multiplying the first equation by a and subtracting the resulting equation from the second one, we obtain $-8y = 8a$, thus $y^0 = -a$. Then, $x^0 = 1$. Thus, the equilibrium point is $(x^0, y^0, z^0) = (1, -a, 0)$.

(b) The solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1(-1)^t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2(-1)^t \begin{pmatrix} -3 \\ a \\ 0 \end{pmatrix} + C_3 5^t \begin{pmatrix} 3 \\ a \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ a \\ 0 \end{pmatrix}$$

where C_1, C_2 and C_3 are arbitrary constants.

4

Consider the ODE

$$(tx^2 + 2tx)\,dt + (2t^2x + 3t^2 + 2x)\,dx$$

(a) (5 points) Show that the above ODE is not exact and find an integrating factor.

Solution:

Let $P(t,x) = tx^2 + 2tx$, $Q(t,x) = (2t^2x + 3t^2 + 2x)$. We have

$$\frac{\partial P}{\partial x} = 2t + 2tx \neq \frac{\partial Q}{\partial t} = 6t + 4tx$$

Hence, the ODE is not exact.

We have

$$\frac{\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x}}{P} = \frac{2}{x}$$

Hence,

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = x^2$$

is an integrating factor of the equation.

(b) (5 points) Find the solution of the following Cauchy problem.

$$tx(x+2)\,dt + (2t^2x + 3t^2 + 2x)\,dx, \quad x(-1) = 1$$

Solution:

Multiplying the ODE by

$$\mu(x) = x^2$$

we obtain that the ODE

$$(2tx^3 + tx^4), dt + \left(3t^2x^2 + 2x^3 + 2t^2x^3\right)\,dx$$

is exact. A potential function is of the form $V = \frac{1}{2}\left(t^2x^4 + 2t^2x^3 + x^4\right)$. So, the general solution is defined implicitly by the equation

$$t^2x^4 + 2t^2x^3 + x^4 = C$$

Plugging in the values $x = 1$, $t = -1$, we obtain $C = 4$. The solution is defined implicitly by the equation

$$t^2x^4 + 2t^2x^3 + x^4 = 4$$

5

Consider the ODE

$$ty' + 4y = 6t^2, \quad y(1) = y_0$$

(a) (5 points) Find the solution for any $y_0 \in \mathbb{R}$.

(b) (5 points) For which value of y_0 do we have that $\lim_{t\to 0+} y(t)$ exist? Is there any value of y_0 such that $\lim_{t\to 0+} y(t) = 1$? Is there any value of y_0 such that $\lim_{t\to 0+} y(t) = -\infty$?

Solution:

The equation is linear. The solution is

$$y(t) = t^2 + \frac{y_0 - 1}{t^4}$$

For $y_0 = 1$ we have $\lim_{t\to 0+} y(t) = 0$. For $y_0 > 1$ we have $\lim_{t\to 0+} y(t) = \infty$. For $y_0 < 1$ we have $\lim_{t\to 0+} y(t) = -\infty$.

6

Consider the differential equation

$$x'' - 6x' - 7x = 16e^{-t} - 14.$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution $x(t)$ of the following initial value problem

$$x'' - 6x' - 7x = 16e^{-t} - 14. \quad x(0) = 7,\ x'(0) = 17.$$

Solution:

$x(t) = 2 - 2e^{-t}t + c_1e^{-t} + c_2e^{7t}$.

$x(t) = 2 - 2e^{-t}t + 2e^{-t} + 3e^{7t}$.