1

Consider a market where only one good is traded where the demand at time t is $D_t = 100 - P_t$ and the supply at time y is $S_t = P_{t-2}$, where P_{t-2} and P_t are the prices at times $t-2$ and t, respectively.

(a) (5 points) Find a difference equation, linear and of order 2, satisfied by the equilibrium prices. *Note:* The equilibrium prices satisfy $D_t = S_t$ for all t.

(b) (5 points) Find the general solution of the difference equation of part (a) above.

(c) (5 points) Find the solution of the difference equation of part (a) above which satisfies the initial conditions $P_0 = 40$, $P_1 = 20$.

(d) (5 points) Fin the maximum and the minimum equilibrium prices.

Solution:

(a) $D_t = S_t$ is equivalent to $P_t + P_{t-2} = 100$.

(b) The characteristic equation is $r^2 + 1 = 0$, with solutions $\pm i$. Hence, the general solution of the homogenous equation is $C_1 \cos(\frac{\pi}{2}t) + C_2 \sin(\frac{\pi}{2}t)$. On the other hand, a particular solution of the complete equation is constant, say A, which must satisfy $A + A = 100$, thus $A = 50$. The general solution of the complete equation of part (a) above is thus

$$P_t = C_1 \cos(\frac{\pi}{2}t) + C_2 \sin(\frac{\pi}{2}t) + 50.$$

(c) We have to solve the two equations

$$\begin{aligned} 40 &= C_1 \cos 0 + C_2 \sin 0 + 50 = C_1 + 50, \\ 20 &= C_1 \cos(\frac{\pi}{2}) + C_2 \sin(\frac{\pi}{2}) + 50 = C_2 + 50, \end{aligned}$$

thus $C_1 = -10$ and $C_2 = -30$. Hence the solution satisfying the initial conditions given is

$$P_t = -10 \cos(\frac{\pi}{2}t) - 30 \sin(\frac{\pi}{2}t) + 50.$$

(d) Note that

$$\begin{aligned} \cos(\frac{\pi}{2}t) &= 1, \text{ and } \sin(\frac{\pi}{2}t) = 0, \text{ for all } t = 0, 4, 8, 12, \dots, \\ \cos(\frac{\pi}{2}t) &= 0, \text{ and } \sin(\frac{\pi}{2}t) = 1, \text{ for all } t = 1, 5, 9, 13, \dots, \\ \cos(\frac{\pi}{2}t) &= -1, \text{ and } \sin(\frac{\pi}{2}t) = 0, \text{ for all } t = 2, 6, 10, 14 \dots, \\ \cos(\frac{\pi}{2}t) &= 0, \text{ and } \sin(\frac{\pi}{2}t) = -1, \text{ for all } t = 3, 7, 11, 15, \dots, \end{aligned}$$

Hence

$$\begin{aligned} P_t &= 40, \text{ for all } t = 0, 4, 8, 12, \dots, \\ P_t &= 20, \text{ for all } t = 1, 5, 9, 13, \dots, \\ P_t &= 60, \text{ for all } t = 2, 6, 10, 14 \dots, \\ P_t &= 80, \text{ for all } t = 3, 7, 11, 15, \dots. \end{aligned}$$

The solution is a cycle of order 4. The maximum price is 80 and the minimum price is 20.

2 Consider the following linear system of difference equations

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & \frac{3}{2} \\ 1 & \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

(a) (5 points) Find the equilibrium point and classify it (unstable, stable, locally asymptotically stable or globally asymptotically stable).

(b) (5 points) Find the general solution of the system.

(c) (5 points) Find the solution of the system which satisfies the initial conditions $x_0 = 0$, $y_0 = 8$ and $z_0 = 2$.

Solution:

(a) The equilibrium points is a constant solution of the system, hence is satisfies

$$\begin{array}{rclllll} x & = & -x & & +\frac{3}{2}z & +1 \\ y & = & x & +\frac{1}{2}y & -z & +3 \\ z & = & -\frac{1}{2}x & & +z & +1. \end{array}$$

The solution is $(x^0, y^0, z^0) = (2, 6, 2)$.

The stability of the equilibrium point depends on the eigenvalues of the system's coefficient matrix

$$A = \begin{pmatrix} -1 & 0 & \frac{3}{2} \\ 1 & \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial is $p_A(\lambda) = (\lambda - \frac{1}{2})(-(1+\lambda)(1-\lambda) + \frac{3}{4})$. The second factor is $\lambda^2 - 1 + \frac{3}{4} = \lambda^2 - \frac{1}{4}$. Thus, the eigenvalues of A are $\frac{1}{2}$, double and $-\frac{1}{2}$, simple. Both are smaller than 1 in absolute value, thus the equilibrium point is globally asymptotically stable.

(b) Let us prove that the matrix A defined above is diaginalizable. We study the rank of the matrix $A - \lambda I_3$ when $\lambda = \frac{1}{2}$. The matrix $A - \frac{1}{2}I_3$ is

$$\begin{pmatrix} -\frac{3}{2} & 0 & \frac{3}{2} \\ 1 & 0 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

which rank is 1. Hence, A is diagonalizable. Let us calculate the eigenvectors.

The eigenspace $S(\frac{1}{2})$ is obtained by solving

$$\begin{pmatrix} -\frac{3}{2} & 0 & \frac{3}{2} \\ 1 & 0 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is clear that $(x, y, x) = x(1, 0, 1) + y(0, 1, 0)$ describe the solutions, thus we take eigenvectors $(1, 0, 1)$ and $(0, 1, 0)$.

The eigenspace $S(-\frac{1}{2})$ is obtained by solving

$$\begin{pmatrix} -\frac{1}{2} & 0 & \frac{3}{2} \\ 1 & 1 & -1 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(The matrix above is $A - (-\frac{1}{2})I_3)$). We get $x = 3z$ and $y = -2z$, thus the solutions are $(3z, -2z, z) = z(3, -2, 1)$, so we choose the eigenvector $(3, -2, 1)$. In consequence, the general solution of the system is:

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1 \left(\frac{1}{2}\right)^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \left(\frac{1}{2}\right)^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 \left(-\frac{1}{2}\right)^t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}.$$

(c) The initial conditions x_0, y_0 y z_0 given in the statement of the exercise determine the value of the constants C_1, C_2 y C_3. Plugging $t = 0$ into the general solution found above, we get the system

$$\begin{pmatrix} 0 \\ 8 \\ 2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}.$$

Solving, $C_1 = 1$, $C_2 = 0$ and $C_3 = -1$. The requested solution is thus:

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \left(\frac{1}{2}\right)^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \left(-\frac{1}{2}\right)^t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}, \qquad t = 0, 1, 2, \ldots.$$

3

Answer the following questions.

(a) (5 points) Find the general solution of the ODE

$$(1+t)x' + x = t(t+1).$$

(b) (5 points) Find the solution of the ODE

$$(1+t)x' + x = t(t+1)$$

which satisfies $x(0) = x(1)$.

Solution:

The ODE is written in the usual form

$$x' + \frac{x}{1+t} = t,$$

for $t \neq -1$.

(a) One of the methods to integrate linear ODEs consists in calculating

$$\mu(t) = e^{\int \frac{1}{t+1}\,dt} = t+1$$

and then, after multiplying the ODE by $\mu(t)$, to obtain

$$\left(x' + \frac{x}{t+1}\right)(t+1) = t(t+1) \quad \Rightarrow \quad (x(t+1))' = t(t+1) \quad \Rightarrow \quad x(t+1) = \int t(t+1)\,dt.$$

After integrating and solving for $x(t)$ we obtain the general solution

$$x(t) = \frac{C}{t+1} + \frac{\frac{t^3}{3} + \frac{t^2}{2}}{t+1}.$$

(b) Imposing $x(0) = x(1)$ with the expression found for the general solution in part (a), we have

$$\frac{C}{t+1} + \frac{\frac{t^3}{3} + \frac{t^2}{2}}{t+1}\Bigg|_{t=0} = \frac{C}{t+1} + \frac{\frac{t^3}{3} + \frac{t^2}{2}}{t+1}\Bigg|_{t=1},$$

that is, $C = \frac{C}{2} + \frac{1}{6} + \frac{1}{4}$, thus $C = \frac{5}{6}$. The solution in question is

$$x(t) = \frac{5}{6(t+1)} + \frac{\frac{t^3}{3} + \frac{t^2}{2}}{t+1}.$$

4

Find the general solution of the ODE

$$x'' - x' = e^{at},$$

where $a \in \mathbb{R}$, in each of the following cases:

(a) (5 points) When $a = 0$.

(b) (5 points) When $a = 1$.

(c) (5 points) When $a \neq 0$ and $a \neq 1$.

Solution:

The homogenous ODE has general solution $C_1 + C_2 e^t$.

(a) When $a = 0$, the ODE is $x'' - x' = 1$. Since constants are solution of the homogenous ODE, a particular solution of the complete ODE is $x_p(t) = At$ with suitable A. The coefficient A is found from $1 = x_p'' - x_p' = -A$, thus $A = -1$. In consequence, the general solution of the complete ODE is given by

$$C_1 + C_2 e^t - t.$$

(b) When $a = 1$, the ODE is $x'' - x' = e^t$. Since e^t is a solution of the homogenous ODE, a particular solution of the complete ODE is $x_p(t) = Ate^t$ with a suitable A. We calculate

$$x_p' = A(t+1)e^t$$
$$x_p'' = A(t+2)e^t.$$

The coefficient A is found from

$$e^t = x_p'' - x_p' = A(t+2)e^t - A(t+1)e^t,$$

which holds for all t iff $A = 1$. In consequence, the general solution of the complete ODE is given by

$$C_1 + C_2 e^t + te^t.$$

(c) When a is not 0 nor 1, the ODE is $x'' - x' = e^{at}$, where e^{at} is not solution of the homogenous ODE. A particular solution of the complete ODE is $x_p(t) = Ae^{at}$ with suitable A. We calculate

$$x_p' = aAe^{at}$$
$$x_p'' = a^2 Ae^{at}.$$

The coefficient A is found from

$$e^{at} = x_p'' - x_p' = a^2 Ae^{at} - aAe^{at},$$

from which $A = 1/(a^2 - a)$. In consequence, the general solution of the complete ODE is given by

$$C_1 + C_2 e^t + \frac{1}{a^2 - a} e^{at}.$$

5

Consider the following system of differential equations

$$\begin{cases} x' &= -2x - y \\ y' &= x - 2y \end{cases}$$

Find and classify the equilibrium point. Skecht the phase diagram.

Solution:

The equilibrium point is $(0,0)$. The characteristic polynomial of the system matrix, $A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$, is $p_A(\lambda) = (2+\lambda)^2 + 1$, which roots are found from $2 + \lambda = \pm\sqrt{-1} = \pm i$. Thus, the roots are complex, $-2+i$ and $-2-i$, both with negative real part. Hence the equilibrium point is globally asymptotically stable and it is an attractive spiral.