1

Consider a market where only one good is traded and that, at time t, the demand is $D = 9 - P_t$ and the supply is $S = P_{t-1} + \frac{1}{4}P_{t-2}$, where P_{t-2}, P_{t-1} and P_t are the prices at times $t-2$, $t-1$ and t, respectively. Find the equilibrium prices P_t.

Hint: The equilibrium prices satisfy $D = S$ at every t.

Solution:

$D = S$ is equivalent to $P_t + P_{t-1} + \frac{1}{4}P_{t-2} = 9$. The general solution of the homogeneous equation is $C_1 2^{-t} + C_2 t 2^{-t}$. A particular solution is $\frac{7}{2}$. Thus

$$P_t = C_1 2^{-t} + C_2 t 2^{-t} + \frac{7}{2}.$$

2

Consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ -2 & 0 & 2a \\ b & -a & 3a \end{pmatrix},$$

where $a > 0$ and $b \in \mathbb{R}$.

(a) (5 points) For what values of the parameters a and b is the matrix A diagonalizable?

(b) (5 points) Calculate the eigenvalues and eigenvectors of A. Write the diagonal form of A and the matrix P when A is diagonalizable.

Solution:

(a) The eigenvalues are $\lambda = a$ double, $\lambda = 2a$ simple. The rank of $A - aI$ is 1 if and only if $b = -2$ (remember that $a \neq 0$). Thus, A is diagonalizable for all $a > 0$ if and only if $b = -2$.

(b) Let $b = -2$. $S(a)$ is generated by $(a, 0, 1)$ and $(0, 1, -\frac{1}{2})$; $S(2a)$ is generated by $(0, 1, 1)$.

3

Consider the following linear system of difference equations,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ -2 & 0 & 2a \\ b & -a & 3a \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with $a > 0$ and $a \neq \frac{1}{2}$, $a \neq 1$. Note that the matrix of the system is the matrix A of Problem 2.

(a) (5 points) Find the equilibrium point and compute all the values of $a > 0$ and $b \in \mathbb{R}$ for which the system is GAS (if any).

(b) (5 points) For the values of a and b for which the matrix is diagonalizable, find the general solution.

Solution:

(a) The vector

$$\frac{1}{1 - 3a + 2a^2} \begin{pmatrix} 0 \\ 2a \\ 1 \end{pmatrix}$$

is the only equilibrium point. Since the eigenvalues are $2a$ and a, with $a > 0$, the equilibrium point is GAS. if and only if $0 < a < \frac{1}{2}$.

(b) A is diagonalizable iff $b = -2$. The solution is

$$C_1 a^t (a,0,1)^T + C_2 (2a)^t (0,1,-\frac{1}{2})^T + C_3 (2a)^t (0,1,1)^T + \frac{1}{1-3a+2a^2}(0,2a,1)^T,$$

where ^{T} indicates transpose.

4

(a) (5 points) Find the general solution of the ODE

$$x' - t^2 x = t^2.$$

(b) (5 points) Find the solution of the ODE

$$x' - t^2 x = t^2$$

which satisfies $x(0) = x(1)$.

Solution:

$x(t) = -1 + Ce^{t^3/3}$. $x(0) = x(1)$ implies $C = e^{-1/3}$, hence $x(t) = -1 + e^{(t^3-1)/3}$.

5

Consider the ODE

$$2e^{2t}dt + (1 + t + 2x)e^{2t}dx = 0.$$

Find the general solution and calculate $\lim_{t\to\infty} x(t)$.

Solution:

e^{-t} is an integrating factor. The solution is found after an integration by parts. From $V_x = 2e^t$ one gets $V = 2xe^t + g(t)$ and then $2xe^t + g'(t) = V_t = e^t(1 + t + 2x)$, from which

$$g'(t) = (1 + t)e^t \;\Rightarrow\; g(t) = te^t.$$

The solution is given by

$$(t + 2x)e^t = C \quad \Rightarrow x(t) = \frac{1}{2}\left(Ce^{-t} - t\right).$$

The limit is $-\infty$.

6

Answer the following questions.

(a) (5 points) Consider the second order differential equation

$$x'' - ax' + bx = te^t,$$

where $a, b \in \mathbb{R}$. Find a and b if it is known that $x(t) = te^t$ is a solution of the equation. Justify your answer.

(b) (5 points) Find the solution of the initial value problem

$$x'' - 2x' + 2x = te^t, \qquad x(0) = 1,\ x'(0) = -1.$$

Solution:

(a) $x' = e^t(t+1)$, $x'' = e^t(t+2)$. Hence

$$(t+2)e^t - a(t+1)e^t + bte^t = te^t$$

and matching coefficients we obtain $a = b = 2$.

(b) $x(t) = e^t(C_1 \cos t + C_2 \sin t) + te^t$ is the general solution. Note that by (a) te^t is a particular solution of the complete ODE, and that the equation $r^2 - 2r + 2 = 0$ has solutions $1 \pm i$.

$x(t) = e^t(\cos t - 3 \sin t) + te^t$ is the particular solution satisfying the initial conditions.