UC3M
TOPICS Advanced Mathematics for Economics
Final Exam JANUARY 21, 2021 **Solutions**

1

Consider the following second order difference equation

$$x_{t+2} + x_{t+1} + \frac{1}{4}x_t = 1.$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution of the initial value problem

$$x_{t+2} + x_{t+1} + \frac{1}{4}x_t = 1, \qquad x_0 = -1, x_1 = 1.$$

(c) (5 points) Consider the solution of the initial value problem

$$x_{t+2} + x_{t+1} + \frac{1}{4}x_t = 1, \qquad x_0 = -1, x_1 = 1,$$

found in the item above. Which is the value of x_3?

Solution:

(a) The characteristic equation is $(r + \frac{1}{2})^2 = 0$, with double root $-\frac{1}{2}$. The general solution of the homogeneous equation is $x_t^h = C_1(-2)^{-t} + C_2 t(-2)^{-t}$. We look for a constant particular solution, $y_t = A$, since 1 is not a solution of the characteristic equation. Plugging this solution into the difference equation we get $A = \frac{4}{9}$. Hence the general solution is

$$x_t = C_1(-2)^{-t} + C_2 t(-2)^{-t} + \frac{4}{9}.$$

(b) $C_1 = -\frac{13}{9}$ and $C_2 = \frac{3}{9}$ $(\frac{1}{3})$. Thus

$$x_t = -\frac{13}{9}(-2)^{-t} + \frac{1}{3}t(-2)^{-t} + \frac{4}{9}.$$

(c) A way is to calculate $x_2 = 1 - \ x_1 - \frac{1}{4}x_0 = 1 - 1 + \frac{1}{4} = \frac{1}{4}$ and $x_3 = 1 - x_2 - \frac{1}{4}x_1 = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$. Another way is by using the solution formula:

$$x_3 = -\frac{13}{9}(-2^{-3}) + \frac{3}{9}(-3 \cdot 2^{-3}) + \frac{4}{9} = \frac{13 - 9 + 32}{9 \cdot 8} = \frac{1}{2}.$$

2

Consider the matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & a & \frac{1}{2} \\ 1 & \frac{1}{2} & a \end{pmatrix},$$

where $a \in \mathbb{R}$.

(a) (5 points) For what values of the parameter a is the matrix A diagonalizable?

(b) (10 points) Calculate the eigenvalues and eigenvectors of A. Write the diagonal form of A and the matrix P when A is diagonalizable.

Solution:

(a) The characteristic polynomial is

$$\left(\frac{1}{2} - \lambda\right)\left((a-\lambda)^2 - \frac{1}{4}\right).$$

Its roots are $\frac{1}{2}$, $a + \frac{1}{2}$, and $a - \frac{1}{2}$. Thus, the eigenvalues of A are:

- $\lambda = \frac{1}{2}$ (multiplicity 2) and $\lambda = -\frac{1}{2}$, if $a = 0$;
- $\lambda = \frac{1}{2}$ (multiplicity 2) and $\lambda = \frac{3}{2}$, if $a = 1$;
- $\lambda = \frac{1}{2}$, $\lambda = a + \frac{1}{2}$, and $\lambda = a - \frac{1}{2}$, all of multiplicity 1, if $a \neq 0$ and $a \neq 1$. In this case the matrix has 3 different eigenvalues, thus A is diagonalizable.

When $a = 0$ or $a = 1$, the rank of $A - \frac{1}{2}I$ is $2 > 1$, thus A is not diagonalizable.

(b) Let $a \neq 0$ and $a \neq 1$.

$$S(\frac{1}{2}) = \langle (2a^2 - 2a, 1, 1 - 2a) \rangle, \quad S(a + \frac{1}{2}) = \langle (0, 1, 1) \rangle, \quad S(a - \frac{1}{2}) = \langle (0, -1, 1) \rangle.$$

Hence

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & a + \frac{1}{2} & 0 \\ 0 & 0 & a - \frac{1}{2} \end{pmatrix}, \quad P = \begin{pmatrix} 2a^2 - 2a & 0 & 0 \\ 1 & 1 & -1 \\ 1 - 2a & 1 & 1 \end{pmatrix}.$$

In the case $a = 0$, in which A is not diagonalizable, $S(\frac{1}{2}) = \langle (0, 1, 1) \rangle$, $S(-\frac{1}{2}) = \langle (0, -1, 1) \rangle$.

In the case $a = 1$, in which A is not diagonalizable, $S(\frac{1}{2}) = \langle (0, -1, 1) \rangle$, $S(\frac{3}{2}) = \langle (0, 1, 1) \rangle$.

3

Consider the following linear system of difference equations,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & a & \frac{1}{2} \\ 1 & \frac{1}{2} & a \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix},$$

with $a \neq \frac{1}{2}$ and $a \neq \frac{3}{2}$. Note that the matrix of the system is the matrix A of Problem 2.

(a) (5 points) Find the equilibrium points.

(b) (5 points) Find all the values of a for which the system is GAS (if any).

(c) (10 points) Find all the values of a for which the system is unstable but there is a stable manifold. Say whether the stable manifold is a line or a plane and if this depends on the values of a. Justify your answers.

Solution:

The matrix associated to the system of difference equations is the matrix A of Problem 2, with $a \neq \frac{1}{2}$ and $a \neq \frac{3}{2}$. The matrix A has eigenvalues $\frac{1}{2}$ and $a \pm \frac{1}{2}$. Note that $a \pm \frac{1}{2} \neq 1$ for all $a \neq \frac{1}{2}$ and $a \neq \frac{3}{2}$.

(a) The equilibrium is $(0, 0, 0)$ since the determinant $|I - A| \neq 0$ because 1 is not an eigenvalue of A.

(b) We know that the eigenvalues are $\frac{1}{2}$, and $a \pm \frac{1}{2}$. Thus, the system is GAS iff both $|a + \frac{1}{2}| < 1$ and $|a - \frac{1}{2}| < 1$. The first inequality is $a \in (-\frac{3}{2}, \frac{1}{2})$ and the second one is $a \in (-\frac{1}{2}, \frac{3}{2})$. The intersection of both intervals is $(-\frac{1}{2}, \frac{1}{2})$, thus the system is GAS iff $|a| < \frac{1}{2}$.

(c) For $|a| > \frac{1}{2}$ the system is unstable but saddle point stable. For $a = -\frac{1}{2}$ the system is not asymptotically stable.

- For $\frac{1}{2} < |a| < \frac{3}{2}$, or $a = -\frac{1}{2}$, two of the eigenvalues are smaller than 1 in absolute value, thus there is a stable manifold of dimension 2, (a plane). The stable manifold is a plane passing through the origin, since that there are two independent eigenvectors associated to eigenvalues smaller than one in absolute value.
- For $|a| > \frac{3}{2}$ or $a = -\frac{3}{2}$, there is only one eigenvalue smaller than 1 in absolute value, thus there is a stable manifold of dimension 1, (a line). The stable manifold is a line passing through the origin, since that there is one eigenvector associated to eigenvalue smaller than one in absolute value.

4

Please, answer the following questions.

(a) (5 points) Find the general solution of the following ODE

$$x'' + x' - 12x = 0$$

(b) (10 points) Find the general solution of the following ODE

$$x'' + x' - 12x = e^{-4t}(2 - 14t)$$

(c) (5 points) Find the solution of the following initial value problem

$$x'' + x' - 12x = e^{-4t}(2 - 14t), \quad x(0) = 0, \quad x'(0) = -7$$

(d) (5 points) Find the solution of the following problem

$$x'' + x' - 12x = e^{-4t}(2 - 14t), \quad \lim_{t\to\infty} x(t) = 0, \quad x(0) = 100$$

Solution:

1. The characteristic equation is r^2+r-12. The roots are -4 and 3. The general solution is $x(t) = Ae^{-4t}+Be^{3t}$.

2. We look for a particular solution of the form $x(t) = t(C + Dt)e^{-4t}$. We have

$$\begin{aligned} x'(t) &= e^{-4t}(-4Ct + C + 2D(1 - 2t)t) \\ x''(t) &= 2e^{-4t}\left(8Ct - 4C + 8Dt^2 - 8Dt + D\right). \end{aligned}$$

So,

$$x'' + x' - 12x = e^{-4t}(2D(1 - 7t) - 7C) = e^{-4t}(2 - 14t).$$

Hence $C = 0$, $D = 1$. The particular solution is $e^{-4t}t^2$. The general solution is

$$x(t) = Ae^{-4t} + Be^{3t} + t^2e^{-4t}.$$

3. From the general solution, we have

$$\begin{aligned} x(0) &= A + B = 0 \\ x'(0) &= 3B - 4A = -7 \end{aligned}$$

The solution is $A = 1$, $B = -1$. The solution is

$$e^{-4t}t^2 + e^{-4t} - e^{3t}$$

4. The limit is finite, only if $B = 0$. Consider $x(t) = Ae^{-4t} + t^2e^{-4t}$ Note that

$$\lim_{t\to\infty} x(t) = 0$$

and $x(0) = A$. Hence, the solution is

$$100e^{-4t} + e^{-4t}t^2$$

5

Consider the ODE

$$3t^2x^2\,dt - e^{\frac{1}{x}}\,dx = 0$$

(a) (5 points) Find the general solution.

(b) (10 points) Find the solution $x(t)$ of the initial value problem

$$3t^2x^2\,dt - e^{\frac{1}{x}}\,dx = 0, \quad x(0) = \frac{1}{\ln 2}.$$

For what values of t is the solution $x(t)$ defined ?

Solution:

(a)

(b) The function

$$\mu(x) = \frac{1}{x^2}$$

is an integrating factor. Multiplying by this integrating factor, we obtain the ODE

$$3t^2\,dt - \frac{1}{x^2}e^{\frac{1}{x}}\,dx = 0$$

We obtain

$$V = t^3 + e^{\frac{1}{x}}$$

and the general solution is given by

$$t^3 + e^{\frac{1}{x}} = C$$

Note: The ODE is also separable.

(c) Plugging in the values $t = 0$, $x = \frac{1}{\ln 2}$ we obtain the equation $e^{\ln 2} = C$. So, $C = 2$. The solution is defined implicitly by

$$t^3 + e^{\frac{1}{x}} = 2$$

Solving for x we have

$$x(t) = \frac{1}{\ln(2 - t^3)}$$

Hence, we need $2 - t^3 > 0$ and $t^3 \neq 1$. That is, $t < 1$. The solution is defined in the interval $(-\infty, 1)$.

6

Consider the differential equation

$$tx' + 2x = \frac{\ln t}{t}, \quad t > 0$$

(a) (5 points) Find the general solution.

(b) (5 points) Find the solution of the following initial value problem

$$tx' + 2x = \frac{\ln t}{t}, \quad x(1) = 0.$$

Solution:

(a) Let $\mu(t) = e^{\int (2/t)dt} = t^2$ and multiply both sides of the equation by μ to obtain

$$(x(t)t^2)' = \frac{\ln t}{\cancel{t^2}}\cancel{t^2},$$

hence, integrating

$$x(t)t^2 = \int \ln t \, dt = t \ln t - t + C,$$

where we have used integration by parts to find the expression of the integral ($u = \ln t$, $dv = dt$). Hence,

$$x(t) = \frac{\ln t - 1}{t} + \frac{C}{t^2}.$$

(b) $0 = x(1) = -1 + C$, thus $x(t) = \frac{\ln t - 1}{t} + \frac{1}{t^2}$.