1

Consider the following second order difference equation

$$x_{t+2} - x_{t+1} + \frac{1}{4}x_t = t.$$

(a) (5 points) Find the general solution.

(b) (5 points) Find a solution of the above equation which satisfies

$$x_0 = 1, \quad x_1 = 0.$$

Solution:

(a) The characteristic equation is $x^2 - x + \frac{1}{4} = 0$, whose unique root is $\frac{1}{2}$. The general solution of the homogeneous equation is $x_t = C_1 2^{-t} + C_2 2^{-t} t$. Now we look for a particular solution of the form $y_t = At + B$. Plugging this solution into the equation we see that $A = 4$ and $B = -16$. Hence, the general solution is

$$x_t = C_1 2^{-t} + C_2 2^{-t} t + 4t - 16.$$

(b) From the system $C_1 - 16 = 1$, $\frac{1}{2}C_1 + \frac{1}{2}C_2 - 12 = 0$ we get $C_1 = 17$ and $C_2 = 7$.

2

Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & b \\ -3 & -1 & 3 \\ b & 0 & 0 \end{pmatrix},$$

where $b \in \mathbb{R}$.

(a) (5 points) For what values of the parameter b is the matrix A diagonalizable?

(b) (5 points) Calculate the eigenvalues and eigenvectors of A **for the case** $\mathbf{b} = \frac{1}{2}$. Write the diagonal form of A and the matrix P.

Solution:

(a) The characteristic polynomial es $-(1+\lambda)(\lambda^2 - b^2)$. Thus, the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = b$, and $\lambda_3 = -b$.

- $b \neq 0$, $b \neq 1$ and $b \neq -1$. A is diagonalizable, since it has three distinct eigenvalues.
- $b = 0$; the eigenvalues are 0, double and -1, simple.
- $b = 1$; the eigenvalues are -1, double and 1, simple.
- $b = -1$; the eigenvalues are -1, double and 1, simple.

In the second case ($b = 0$), $\operatorname{rank}(A - 0I_3) = \operatorname{rank} \begin{pmatrix} 0 & 0 & 0 \\ -3 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = 1$. A is diagonalizable.

In the third case ($b = 1$), $\operatorname{rank}(A - (-1)I_3) = \operatorname{rank} \begin{pmatrix} 1 & 0 & 1 \\ -3 & 0 & 3 \\ 1 & 0 & 1 \end{pmatrix} = 2$. A is not diagonalizable.

In the fourth case ($b = -1$), $\operatorname{rank}(A - (-1)I_3) = \operatorname{rank} \begin{pmatrix} 1 & 0 & -1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{pmatrix} = 1$. A is diagonalizable.

(b) When $b = \frac{1}{2}$, the eigenvalues are $-1, \pm\frac{1}{2}$, thus A is diagonalizable and it is easy to compute

$$S(-1) = \langle(0,1,0)\rangle, \quad S(\frac{1}{2}) = \langle(1,0,1)\rangle, \quad S(-\frac{1}{2}) = \langle(-1,12,1)\rangle.$$

Hence

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}$$

3

Consider the following linear system of difference equations,

$$\begin{aligned} x_{t+1} &= \frac{z_t}{2} + 3 \\ y_{t+1} &= -3x_t - y_t + 3z_t + 2 \\ z_{t+1} &= \frac{x_t}{2} + 3. \end{aligned}$$

Note that the matrix of the system is the matrix A of Problem 2, when $b = \frac{1}{2}$ (and thus A is diagonalizable).

(a) (5 points) Compute the equilibrium point and study its stability.

(b) (5 points) Find the general solution of the above linear system of difference equations.

Solution:

The matrix associated to the system of difference equations is the matrix A of Problem 2, when $b = \frac{1}{2}$, thus it is diagonalizable.

(a) The equilibrium satisfies

$$\begin{aligned} x &= \frac{z}{2} + 3 \\ y &= -3x - y + 3z + 2 \\ z &= \frac{1}{2}x + 3, \end{aligned}$$

thus it is the point $(6, 1, 6)$. We have showed in Problem 2 above that the eigenvalues of the system are -1, $\frac{1}{2}$ and $-\frac{1}{2}$. Since that not every eigenvalue is smaller than one in absolute value, the system is not asymptotically stable.

(b) The system is diagonalizable; the solution is

$$X_t = C_1\lambda_1^t \mathbf{v}_1 + C_2\lambda_2^t \mathbf{v}_2 + C_3\lambda_3^t \mathbf{v}_3 + X^0,$$

where $X = (x, y, z)'$, C_1, C_2, C_3 are constants, λ_i are eigenvalues and $\mathbf{v}_i$ are the corresponding column eigenvectors, for $i = 1, 2, 3$, and X^0 is the equilibrium. In this case, where $\lambda_1 = -1$, $\lambda_2 = \frac{1}{2}$ and $\lambda_3 = -\frac{1}{2}$, the general solution is

$$X_t = C_1(-1)^t(0, 1, 0)' + C_2\frac{1}{2^t}(1, 0, 1)' + C_3\left(-\frac{1}{2}\right)^t(-1, 12, 1)' + (6, 1, 6)'.$$

(the prime denotes column vectors)

4

Consider the following ordinary differential equation

$$x(x-t)\,dt + \left(3xt - t^2 + \frac{1}{x^2}\right)\,dx = 0.$$

(a) (7 points) Find the general solution.

(b) (3 points) Find the solution that satisfies $x(0) = e^2$.

Solution:

(a) The ODE $P(t,x)\,dt + Q(t,x)\,dx = 0$ is not exact, since

$$P_x = 2x - t \neq 3x - 2t = Q_t.$$

An integrating factor independent of t is $\mu(x) = e^{\int \frac{Q_t - P_x}{P}\,dx}$, since

$$\frac{Q_t - P_x}{P} = \frac{3x - 2t - (2x - t)}{x(x-t)} = \frac{x - t}{x(x-t)} = \frac{1}{x}$$

is independent of t. Thus, $\mu(x) = x$ makes the equation exact. Let

$$x^2(x-t)\,dt + \left(3x^2t - xt^2 + \frac{1}{x}\right)\,dx = 0$$

the new ODE obtained. A potential function is $V(t,x) = \int x^2(x-t)dt = x^3t - \frac{x^2t^2}{2} + g(x)$; to find g, use $V_x = 3x^2t - xt^2 + \frac{1}{x}$ to get the identity

$$3x^2t - xt^2 + g'(x) = 3x^2t - xt^2 + \frac{1}{x}.$$

Hence $g(x) = \ln x$. The general solution is thus given by

$$x^3t - \frac{x^2t^2}{2} + \ln x = C, \quad C \text{ constant.}$$

(b) $x^3t - \frac{x^2t^2}{2} + \ln x = 2$

5

Consider the differential equation

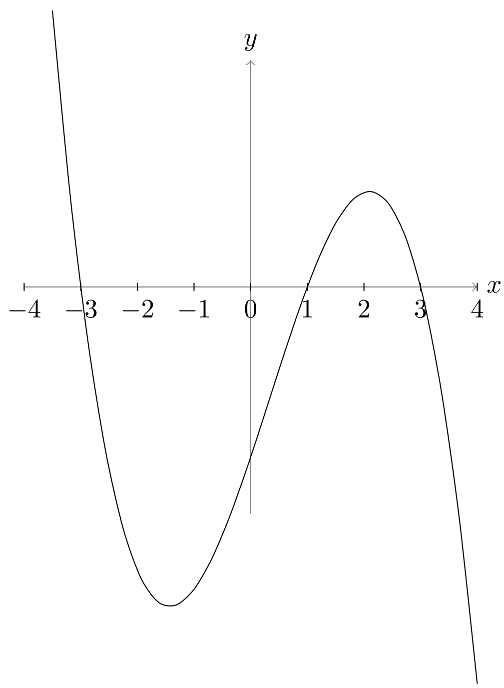
$$x' = \left(1 + \frac{x}{3}\right)(1 - x)(x - 3)$$

(a) (5 points) Find the equilibrium points, draw the phase diagram of the differential equation and study the stability of its equilibrium points.

(b) (5 points) Draw the graph of the solution $x(t)$ of the differential equation, which starts at $x(0) = 2$ and find the limits $\lim_{t\to\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.

Solution:

(a) The equilibrium points are -3, 1 y 3.



Clearly, -3 is unstable, 0 locally asymptotically stable 3 is unstable.

(b) $\lim_{t\to\infty} x(t) = 1$ and $\lim_{t\to-\infty} x(t) = 3$.

6

Consider the differential equation

$$x'' + 3x' + 2x = e^{-t}.$$

(a) (5 points) Find the general solution.

(b) (5 points) Calculate $\lim_{t\to\infty} x(t)$ for any solution $x(t)$ found in part (a) above.

Solution:

(a)

$$x(t) = C_1 e^{-t} + C_2 e^{-2t} + te^{-t}.$$

(b) The terms e^{-t} and e^{-2t} tends to zero. The term $te^{-t} = \frac{t}{e^t}$ converges to the same limit than $\frac{1}{e^t}$ by l'Hopital. Hence every solution tends to zero.