1

(a) (5 points) Find the general solution of the following difference equation

$$x_{t+2} + 2x_{t+1} - 3x_t = 0.$$

(b) (5 points) Find the general solution of the following difference equation

$$x_{t+2} + 2x_{t+1} - 3x_t = 16.$$

(c) (5 points) Find the particular solution of the following difference equation.

$$x_{t+2} + 2x_{t+1} - 3x_t = 16, \quad x_0 = 4, x_1 = 8.$$

Solution:

(a) The characteristic equation is $r^2 + 2r - 3 = 0$ whose solutions are $r_1 = 1$, $r_2 = -3$. The general solution is

$$x_t = A + B(-3)^t.$$

(b) Since $r_1 = 1$ is a root of the characteristic equation, we look for a particular solution of the form $y_t = At$. Plugging into the difference equation the values

$$y_t = At, \quad y_{t+1} = At + A, \quad y_{t+2} = At + 2A$$

and solving for A we obtain $A = 4$. The general solution is

$$x_t = 4t + A + B(-3)^t.$$

(c) We have that the general solution is of the form $x_t = 4t + A + B(-3)^t$. Thus, $x_0 = A + B$, $x_1 = A - 3B + 4$. We solve the system

$$\begin{aligned} A + B &= 4 \\ A - 3B + 4 &= 8 \end{aligned}$$

and we obtain the solution $A = 4$, $B = 0$. Thus, the solution of the difference equation is

$$x_t = 4t + 4.$$

2

Consider the matrix

$$A = \begin{pmatrix} a & 0 & 0 \\ 1 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

where $a \in \mathbb{R}$.

(a) (10 points) For what values of the parameter a is the matrix A diagonalizable?

(b) (10 points) For the values of parameter a for which the matrix A diagonalizable, write its diagonal form and the matrix P.

Solution:

(a) The characteristic polynomial of A is $(a-\lambda)\big(-\lambda(1-\lambda)-2\big)$, with roots

$$\lambda_1 = a,\ \lambda_2 = 2,\ \lambda_3 = -1.$$

If $a \neq 2$ and $a \neq -1$, then the polynomial has three different roots, thus the matrix A is diagonalizable. In the case $a = 2$, $\lambda_1 = \lambda_2 = 2$ is double. The rank of $A - 2I$ is

$$\operatorname{rank}(A-2I) = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix} = 2 \neq 1,$$

thus A is not diagonalizable. In the case $a = -1$, $\lambda_1 = \lambda_3 = -1$ is double. The rank of $A + I$ is

$$\operatorname{rank}(A+I) = \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} = 2 \neq 1,$$

thus A is not diagonalizable.

(b) A is diagonalizable if and only if $a \neq 2$ and $a \neq -1$.

- To find an eigenvector associated to a, we solve $(A - aI)u = 0$, that is

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -a & -2 \\ 0 & -1 & 1-a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  and obtain $y = (1-a)z$, $x = (2 + a(1-a))z$.

- To find an eigenvector associated to 2, we solve $(A - 2I)u = 0$, that is

$$\begin{pmatrix} a-2 & 0 & 0 \\ 1 & -2 & -2 \\ 0 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  and obtain $x = 0$, $y = -z$.

- To find an eigenvector associated to -1, we solve $(A + I)u = 0$, or

$$\begin{pmatrix} a+1 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

  and obtain $x = 0$, $y = 2z$.

Suitable matrices D and P are

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 + a(1-a) & 0 & 0 \\ 1-a & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

respectively.

3

Consider the following non-homogeneous system of difference equations $X_{t+1} = AX_t + B$, where the matrix of coefficients is the matrix A of Problem 2 above with $a = \frac{1}{2}$, given by

$$\begin{aligned} x_{t+1} &= \frac{x_t}{2} + 1 \\ y_{t+1} &= x_t - 2z_t - 2 \\ z_{t+1} &= -y_t + z_t + 2 \end{aligned}$$

(a) (10 points) Find the equilibrium point of the system of difference equations and study its stability.

(b) (5 points) Compute the general solution of the above system of difference equations.

Solution:

(a) The equilibrium point is found by solving

$$\left\{ \begin{array}{rcl} x &=& \frac{x}{2} + 1 \\ y &=& x - 2z - 2 \\ z &=& -y + z + 2 \end{array} \right. ,$$

which solution is $x^0 = 2$, $y^0 = 2$, $z^0 = -1$. By Problem 2, the eigenvalues of the matrix of the system are $\frac{1}{2}$, 2 and -1, thus the system is unstable. Since one of the eigenvalues is smaller than 1 in absolute value, there are initial conditions for which the solutions converges to the equilibrium point, which is a saddle point.

1. We know that the system is diagonalizable and we have found the eigenvalues and eigenvectors in Problem 2. The general solution is

$$\begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = C_1 2^{-t} \begin{pmatrix} 2 + \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix} + C_2 2^t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 (-1)^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix},$$

where C_1, C_2 and C_3 are constants.

4

(a) (10 points) Find the general solution of the following ODE

$$\ddot{x} + 4x = 16\,te^{2t}$$

(b) (5 points) Find the solution $x(t)$ of the above ODE that satisfies the following initial conditions

$$x(0) = 0,\ \dot{x}(0) = -2$$

Solution:

(a) The characteristic equation is $r^2 + 4 = 0$ whose roots are $\pm 2i$. Hence, the general solution of the associated homogeneous equation is

$$x^h(t) = C_1 \sin(2t) + C_2 \cos(2t).$$

We look now for a particular solution of the form

$$x(t) = (At + B)e^{2t}$$

Thus,

$$x'(t) = e^{2t}(2At + A + 2B)$$
$$x''(t) = 4e^{2t}(At + A + B)$$
$$x'' + 4x = 4e^{2t}(2At + A + 2B)$$

and we obtain $4A + 8B = 0$ and $8A = 16$. Solving, we find $A = 2$, $B = -1$. Hence, the general solution is

$$x^g(t) = C_1 \sin(2t) + C_2 \cos(2t) + (2t - 1)e^{2t}$$

(b) Note that plugging the values $x^g(0) = C_1 - 1 = 0$ and $\dot{x}^p(0) = 2C_2 = -2$ into the general solution, we get $C_1 = 1$ and $C_2 = -1$. Hence, the solution is

$$x^p(t) = \sin(2t) - \cos(2t) + (2t - 1)e^{2t}$$

5

Consider the following system of ODE's

$$\begin{aligned} x' &= -x + 3y - 4 \\ y' &= 3x - y + 1 \end{aligned}$$

(a) (10 points) What is the equilibrium point of the system. Is it stable? In the case that the equilibrium point is not stable, study whether the stable manifold exists and find it.

(b) (5 points) Compute the general solution.

Solution:

(a) The equilibrium point is $(1, 2)$. The eigenvalues are 2 and -4, so the equilibrium point is a saddle point. The stable manifold is $S(-4) =< (1, -1) >$.

(b) $S(2) =< (1, 1) >$. The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 1 \\ 11 \end{pmatrix}.$$

6

Consider the following differential equation

$$\left(\frac{y}{x}\right) dx + \left(1 + \frac{e^y}{x}\right) dy = 0.$$

(a) (5 points) Show that it has an integrating factor of the form $\mu(x)$.

(b) (10 points) Compute the general solution of the above DE.

(c) (5 points) Compute the particular solution of the following initial value problem.

$$\left(\frac{y}{x}\right) dx + \left(1 + \frac{e^y}{x}\right) dy = 0, \quad y(1) = 0.$$

Solution:

(a) Letting $P(x, y) = \frac{y}{x}$ and $Q(x, y) = 1 + \frac{e^y}{x}$, we have that

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{\frac{1}{x} - \left(\frac{-e^y}{x^2}\right)}{1 + \frac{e^y}{x}} = \frac{1}{x} \frac{\left(1 + \frac{e^y}{x}\right)}{\left(1 + \frac{e^y}{x}\right)} = \frac{1}{x},$$

is independent of y. Hence, the DE has an integrating factor $\mu(x)$.

(b) Let

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$$

Multiplying the DE by x, it becomes exact. Let

$$V(x, y) = \int x Q(x, y) dy = \int (x + e^y)\, dy = xy + e^y + g(x).$$

Since $\frac{\partial V}{\partial x} = xP = y$, we find $y + g'(x) = y$, hence $g(x) = 0$. The general solution of the DE is then

$$xy + e^y = C.$$

(c) Plugging $y = 0$ and $x = 1$ into the general solution found in part (b) above, we get $C = 1$. Hence the particular solution is

$$xy + e^y = 1.$$