1.

Find the solution of $x_{t+2} - x_{t+1} + \dfrac{1}{2}x_t = t$ with $x_0 = 0$, $x_1 = 0$.

Solution:

The general solution is the sum of the general solution of the homogeneous equation and one particular solution. The characteristic equation is $r^2 - r + \frac{1}{2} = 0$, with complex solutions $r_{1,2} = \frac{1}{2} \pm i\frac{1}{2}$, with module $\sqrt{\frac{1}{2^2} + \frac{1}{2^2}} = \frac{1}{\sqrt{2}}$ and argument $\arctan\left(\frac{1}{2}/\frac{1}{2}\right) = \arctan 1 = \frac{\pi}{4}$. Thus, the general solution of the homogeneous equation is $2^{-t/2}(A\cos t\frac{\pi}{4}t) + B\sin\left(\frac{\pi}{4}t\right))$, $A, B \in \mathbb{R}$. Now try a particular solution of the form $at + b$. To find a, b, plug this expression into the equation to get

$$a(t+2) + b - a(t+1) - b + \frac{1}{2}(at+b) = t$$

thus, $a = 2$ and $b = -4$. The general solution of the complete equation is

$$x_t = 2^{-t/2}\left(A\cos\left(\frac{\pi}{4}t\right) + B\sin\left(\frac{\pi}{4}t\right)\right) + 2t - 4.$$

To find the particular solution passing through $x_0 = 0$, $x_1 = 0$ we impose

$$\begin{aligned} 0 &= A - 4, \\ 0 &= \tfrac{1}{\sqrt{2}}(A\tfrac{\sqrt{2}}{2} + B\tfrac{\sqrt{2}}{2}) + 2 - 4. \end{aligned}$$

The solution of this system is $A = 4$ and $B = 0$. Hence, the final answer is

$$x_t = 4 \cdot 2^{-t/2}\cos\left(\frac{\pi}{4}t\right) + 2t - 4.$$

2.

Find the solution of the linear system of difference equations

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

satisfying initial conditions $x_0 = 0$, $y_0 = 0$.

Solution:

Let us see if the matrix A of the system is diagonalizable.

$$|A - \lambda I| = -(\lambda(1-\lambda) - 2 = 0 \Rightarrow \lambda = -1 \text{ or } \lambda = 2.$$

Thus the matrix is diagonalizable since the eigenvalues are distinct. Now we compute a pair of independent eigenvectors by solving $(A - \lambda I)\mathbf{u} = \mathbf{0}$. For $N(-1)$ we solve

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that gives $u = -2v$, thus we can choose $(-2, 1)$. For $N(2)$ we solve

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

thus we take $(1, 1)$. The general solution of the homogeneous system is thus

$$a(-1)^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + b2^t \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where a, b are arbitrary constants. A particular solution is a constant vector $\begin{pmatrix} u \\ v \end{pmatrix}$. Plugging this vector into the system we find the following linear system

$$\begin{cases} u &= 2v + 2, \\ v &= u + v - 2. \end{cases}$$

Solving we get $u = 2$, $v = 0$. Thus, the general solution of the complete system is

$$a(-1)^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} + b2^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

Now we employ the initial values to determine the constants a and b. Letting $t = 0$ in the general solution we have

$$a \begin{pmatrix} -2 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We find $a = 2/3$ and $b = -2/3$.

Note that we could also used more directly the formula

$$X^0 + PD^tP^{-1}(X_0 - X^0),$$

where $X^0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $X_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

3.

(a) (5 points) Solve $x'(t) = \dfrac{t+2}{x+1}$.

(b) (5 points) Solve $(x^2 - xe^t)dt + (2tx - e^t)dx = 0$, $x(0) = 10$.

Solution:

(a) $x'(x+1) = t+2$; $\int (x+1)dx = \int (t+2)dt$; $\dfrac{x^2}{2} + x = \dfrac{t^2}{2} - C$, with C constant.

(b) The equation is exact since $(x^2 - xe^t)_x = 2x - e^t = (2tx - e^t)_t$

$$V(t,x) = \int (x^2 - xe^t)dt = x^2 t - xe^t + g(x)$$

and

$$V_x(t,x) = 2xt - e^t + g'(x) = 2tx - e^t,$$

so $g \equiv C$ is constant. The solution is implicitly given by

$$x^2 t - xe^t + C = 0.$$

At $t = 0$ we have $-10 + C = 0$, thus finally $x^2 t - xe^t + 10 = 0$.

4.

Find the the equilibrium points of the nonlinear differential equation

$$x'(t) = ax(t) - x^2(t),$$

where $a \in \mathbb{R}$ is a parameter. By using phase diagrams, study whether the equilibrium points are locally asymptotically stable depending on the values of a.

Solution:

Let $f(x) = x(a - x)$. It is a concave parabola. Fixed points are given by $f(x) = 0$, so we get $x_1 = 0$ and $x_2 = a$. Case (i) $a > 0$. Then f is negative in $(-\infty, 0)$ and (a, ∞) and positive in $(0, a)$. Thus, $x' < 0$ in the two former intervals and $x' > 0$ ion the latter interval and hence x_1 is unstable and x_2 is l.a.s. Case (ii) $a < 01$. Then f is negative in $(-\infty, a)$ and $(0, \infty)$ and positive in $(a, 0)$. Thus, $x' < 0$ in the two former intervals and $x' > 0$ ion the latter interval and hence, x_1 is l.a.s. and x_2 is unstable. Case (iii) $a = 0$. Clearly 0 is unstable.

5.

The demand and supply functions for a given commodity are

$$Q_d(t) = 100 - 2P(t) + \dot{P}(t),$$
$$Q_s(t) = -50 + P(t)$$

respectively, where $P(t)$ is the price of the commodity at time $t \geq 0$. The price evolves according to

$$\dot{P}(t) = \frac{1}{6}(Q_d(t) - Q_s(t)), \quad P(0) = 10.$$

(a) (5 points) Find a first order ODE for the price $P(t)$ and the equilibrium price, P^0.

(b) (5 points) Find the time path of the price $P(t)$. Is it convengent?

Solution:

(a) By plugging Q_d and Q_s into the equation for $\dot{P}$ and simplifying we get $\dot{P} = 30 - \frac{3}{5}P$. The equilibrium point, $\dot{P} = 0$, is $P^0 = 50$.

(b) The solution is $P(t) = 50 + 10e^{-3/5t}$. Yes, it converges to 50.