ADVANCED MATHEMATICS FOR ECONOMICS – 2014/2015

Sheet 3. Difference Equations (2)

3-1. Find the solutions of the equations (a) $x_{t+2} - \frac{1}{4}x_t = \sin\frac{\pi}{2}t$, $x_0 = 1/2$, $x_1 = 0$; (b) $x_{t+2} - x_{t+1} + x_t = e^{-t} + 1$.

3-2. Investigate the stability of the following equations: (a) $x_{t+1} - \frac{1}{4}x_t = b_t$, (b) $x_{t+2} - x_{t+1} + x_t = c_t$, where $\{b_t\}$ and $\{c_t\}$ are given sequences.

3-3. Solve the Fibonacci equation $x_{t+2} = x_{t+1} + x_t$, $x_0 = x_1 = 1$ and check that

$$\lim_{t\to\infty} \frac{x_{t+1}}{x_t} = \frac{1+\sqrt{5}}{2} \equiv \varphi, \quad \text{the golden section.}$$

3-4. Consider the equation obtained in the multiplier–accelerator model of growth studied in the class notes,

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b,$$

with $a > 0$, $c > 0$ and $a \neq 1$.

(a) Find a particular solution of this equation;
(b) Discuss whether the solutions of the characteristic equation are real or complex.
(c) Find the general solution in each of the following cases.
  (i) $a = 4$, $c = 1$;
  (ii) $a = \frac{3}{4}$, $c = 3$;
  (iii) $a = 0.5$, $c = 1$.

3-5. Let C_t denotes consumption, K_t capital stock, Y_t net national product. We suppose that these variables are related as

$$\begin{aligned} C_t &= cY_{t-1}, \\ K_t &= \sigma Y_{t-1}, \\ Y_t &= C_t + K_t - K_{t-1}, \end{aligned}$$

where c and σ are positive constants.

(a) Give an economic interpretation of the equations.
(b) Derive a second order difference equation for Y_t.
(c) Find necessary and sufficient conditions for the solution of the equation in (b) to have explosive oscillations.

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Sheet 4. Difference Equations (3)

4-1. Prove the equivalence of the two following assertions:
(a) The quadratic equation $\lambda^2 - p\lambda + q = 0$ has roots satisfying $|\lambda| < 1$;
(b) $|p| < 1 + q$ and $q < 1$ (*Jury condition*).

4-2. Find the interval of values of the parameter a for which the system

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} a & -\frac{1}{2} \\ 2 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}$$

is globally asymptotically stable. Find the stable manifold (if possible) for the cases $a = -5/3$ and $a = 5/2$.

4-3. Suppose that a firm starts activity with 100 machines of the same age. After 2 years, machines become obsolete and must be replaced for a new one. Moreover, it is known that 11% of the machines of 1 year will fail and must be also replaced. Write down the equations of the dynamical system involved and the initial condition.

4-4. Consider the following deterministic version of a model of inequality transmission across generations of Gary Solon[1]. Parents invest I_{t-1} into the child's human capital h_t, with effect $h_t = \theta \ln I_{t-1} + g_t$, where $\theta > 0$ and g_t is human endowment the child receives independently of parent's investment (genetic inheritance). Assume that $g_t = \delta + \lambda g_{t-1}$, $0 < \lambda < 1$. Lifetime income of the child is given by $\ln y_t = \mu + p h_t$.
(a) Interpret the coefficients θ, λ and p.
(b) Assuming that parents' investment is a positive constant fraction of their income, $I_{t-1} = e^k y_{t-1}$, $k \leq 0$, write down a second order difference equation for $\ln y_t$.
(c) Show that in this model the log of income of the child is positively related with the parents' log of income, but inversely related with the grandfather's log of income.
(d) Show that the model never shows an oscillating behavior.
(e) Find the general solution when $\lambda = \gamma$ and show in this case that the income converges to an equilibrium.

4-5. K is a student with the following habit: once she studies one day, it is likely that she will not study the following day with probability 0.7. On the other hand, the probability that she does not study two consecutive days is 0.6. Assuming that today K has promised to study, with which probability does K study in the long run?

4-6. A psychologist places a mouse inside a jail with two doors, A and B. Going through door A, the mouse receive an electrical shock. The mouse never chooses door A twice in a row. Some food is behind door B. After choosing B, the probability of returning to B in the following day is 0.6. At the beginning of the experiment (Monday), the mouse chooses A or B with the same probability.
(a) With which probability does the mouse choose door A on Thursday?
(b) Which is the stationary distribution of this experiment?
(c) What do you think the mouse thinks about the psychologist?

[1]Gary Solon. "Theoretical models of inequality transmission across multiple generations". (18790), February 2013.

4-7. **Phillips curve I**.

The *Phillips curve* relates negatively the rate of growth of money wage w and the unemployment rate U,

$$w = f(U), \qquad f'(U) < 0. \tag{1}$$

This was justified empirically by A.W. Phillips for the U.K. in a very influential paper[2]. Later, the relation was postulated to affect also to the rate of inflation, p, since a growing–money wage costs would had inflationary effects[3],

$$p = w - T.$$

Here, T denotes an exogenous increase in labor productivity (hence inflation appears only if the salary grows faster than productivity). Assuming a linear form of function f, $f(U) = \alpha - \beta U$, we will have that at every $t \geq 1$

$$p_t = \alpha - T - \beta U_t, \qquad \alpha, \beta > 0. \tag{2}$$

On the other hand, the theory links the unemployment rate and the rate of inflation according to

$$U_{t+1} - U_t = -k(m - p_t), \qquad 0 < k \leq 1, \tag{3}$$

where m is the rate of growth of the nominal money balance[4]. Noticing that $m - p$ is the rate of growth of real money, Eqn. (3) establishes that the rate of growth of unemployment is negatively related with the rate of growth of real money.

Find a difference equation for U_t and study the stability properties of the solution.

4-8. **Phillips curve II**.

Continuing with the Phillips' model, we analyze now the modification introduced by Friedman[5], considering the *expected–augmented* version of the Phillips relation

$$w = f(U) + g\pi, \qquad (0 < g \leq 1), \tag{4}$$

where π denotes the expected rate of inflation. The idea is that if an inflationary trend has been observed long enough, people form certain inflation expectations, which they attempt to incorporate into their money–wage demands. Then, (2) results in the equation

$$p_t = \alpha - T - \beta U_t + g\pi_t, \quad t \geq 0. \tag{5}$$

How is formed inflation expectations? Commonly is is assumed the *adaptive expectations* hypothesis

$$\pi_{t+1} - \pi_t = j(p_t - \pi_t), \qquad 0 < j \leq 1. \tag{6}$$

This means that when the actual rate of inflation p turns out to exceed the expected rate π, the latter, having now been proven to be too low, is revised upward. Conversely, if p falls short of π, then π is revised in the downward direction. The speed of adjustment is j.

Consider the model given by Eqs. (3), (5) and (6).

(a) Eliminate p_t and write a system of linear difference equations for the variables (U_t, π_t).
(b) Using the Jury condition, determine whether the system is g.a.s.
(c) Find and interpret the fixed or equilibrium points of the system.

[2] A.W. Phillips (1956) "The relationship between unemployment and the rate of change of money wage rates in the United Kingdom," *Economica*, November 1958, pp. 283–299.

[3] The rate of growth of money wage is $(W_{t+1} - W_t)/W_t$, where W_t is wage at time t; the rate of inflation is the rate of the general price level, $p = (P_{t+1} - P_t)/P_t$.

[4] That is, $m = (M_{t+1} - M_t)/M_t$, where M_t is the nominal money balance, fixed by the monetary authority. It is supposed here that m constant, independent of t.

[5] M. Friedman (1968) "The role of monetary policy," *American Economic Review*, pp. 1–17.