TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

Sheet 8. Differential Equations (4)

8-1. Classify the equilibrium point $(0,0)$ of the following systems, in terms of the parameter α.

(a) $\dot{X} = \begin{pmatrix} \alpha & 0 \\ 6 & 2\alpha \end{pmatrix} X, \quad (\alpha \neq 0).$

(b) $\dot{X} = \begin{pmatrix} \alpha & -3 \\ 3 & \alpha \end{pmatrix} X.$

8-2. Find and classify the equilibrium point of the following systems. In the case of a saddle, find the stable manifold.

(a) $\dot{X} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X + \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$

(b) $\dot{X} = \begin{pmatrix} 2 & -5 \\ 5 & -6 \end{pmatrix} X + \begin{pmatrix} 1 \\ 9 \end{pmatrix}.$

8-3. Study the stability of the following systems.

$$\text{a)} \begin{cases} \dot{x} &= e^x - 1, \\ \dot{y} &= ye^x. \end{cases} \qquad \text{b)} \begin{cases} \dot{x} &= x^3 + 3x^2 y + y, \\ \dot{y} &= x(1+y^2). \end{cases}$$

8-4. The model of Obst[1] of monetary policy in the presence of an inflation adjustment mechanism is as follows. The quotient M_d/M_s (money demand/money supply), is denoted by μ; $p = \dot{P}/P$ is the inflation rate (P is the price level of the economy); $q = \dot{Q}/Q$ the constant (exogenous) rate of growth of GDP, Q, and $m = \dot{M}_s/M_s$ the monetary expansion rate. The evolution of p follows the Walrasian adjustment mechanism

$$\dot{p} = h(1-\mu), \qquad 0 < h < 1 \text{ a parameter.}$$

Hence an excess in the monetary supply $M_s > M_d$, leads to a positive increment in the inflation rate. To stipulate the time evolution of μ we consider the following assumption: monetary demand is proportional to GDP in nominal terms, that is,

$$M_d = aPQ, \qquad a > 0 \text{ constant,}$$

hence

$$\mu = a\frac{PQ}{M_s}.$$

Taking logarithms

$$\ln \mu = \ln a + \ln P + \ln Q - \ln M_s,$$

and taking the derivative with respect to time we get

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} - \frac{\dot{M}_s}{M_s} = p + q - m.$$

Hence, the system of ODEs in the model of Obst is

$$\dot{p} = h(1-\mu),$$
$$\dot{\mu} = (p+q-m)\mu.$$

The exercise studies the effect of the monetary policy chosen by the central bank, given by m.

(a) Suppose that $m = \overline{m}$ is constant (*exogenous and constant monetary expansion rate*) and that $\overline{m} > q$. Show that the system has a center.

[1] N. P. Obst (1978) "Stabilization policy with an inflation adjustment mechanism". *Quarterly Journal of Economics*, May, pp. 355–359.

(b) Suppose that $m = \overline{m} - \alpha p$ with $\alpha > 0$ (*countercyclical conventional monetary policy*) and $\overline{m} > q$. Show by means of the phase portrait that the qualitative behavior of the system is similar to (a) above.

(c) Suppose that $m = \overline{m} - \alpha \dot{p}$ (*Obst's Rule*) with $\alpha > 0$ and $\overline{m} > q$. Prove that for some values of α the system has a spiral attractor.

(d) What do you think about the stabilization properties of the countercyclical rule and Obst's Rule?