## TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

Sheet 7. Differential Equations (3)

Solutions

7-1. *Answer the following questions:*

(a) *Form the homogeneous linear ODEs from their characteristic equation.*

(i) $r^2 - 3r + 5 = 0$;

(ii) $r(r+2) = 0$.

(b) *Form the homogeneous linear ODEs from the roots of their characteristic equation.*

(i) $r_1 = 1$, $r_2 = 4$;

(ii) $r_1 = 3 - 4i$, $r_2 = 3 + 4i$.

(c) *Form the homogeneous linear ODEs from their general solution.*

(i) $C_1 e^t + C_2 e^{-2t}$;

(ii) $C_1 e^{-2t} + C_2 t e^{-2t}$;

(iii) $e^{-t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

**Solution:**

(a) (i) $\ddot{x} - 3\dot{x} + 5x = 0$.

(ii) $\ddot{x} + 2\dot{x} = 0$.

(b) (i) $(r-1)(r-4) = r^2 - 5r + 4 \Rightarrow \ddot{x} - 5\dot{x} + 4x = 0$.

(ii) $(r-(3+4i))(r-(3-4i)) = r^2 - (3+4i)r - (3-4i)r + (3+4i)(3-4i) = r^2 - 6r + 25 \Rightarrow \ddot{x} - 6\dot{x} + 25x = 0$.

(c) (i) The roots are 1 and 2, thus the characteristic equation is $(r-1)(r+2) = 0$, from which $\ddot{x} + \dot{x} - 2x = 0$.

(ii) The root is 2 (double), thus the characteristic equation is $(r-2)^2 = 0$, from which $\ddot{x} - 4\dot{x} + 4x = 0$.

(iii) The roots are complex conjugates, with real part $-1/2$ and imaginary part 1, thus the characteristic equation is

$$(r - (-\frac{1}{2} + i))(r - (-\frac{1}{2} - i)) = r^2 + r + \frac{5}{4} = 0 \Rightarrow \ddot{x} + \dot{x} + \frac{5}{4}x = 0.$$

7-2. *Find the solution of the following equations.*

(a) $x'' - ax = t$, *where* $a \in \mathbb{R}$,

(b) $x'' - 2x + x = \sin t$, $x(0) = \dot{x}(0) = 1$.

(c) $x'' - 3x' + 2x = (t^2 + t)e^{3t}$,

**Solution:**

(a) Let us consider first the special case $a = 0$. The equation reduces to $\ddot{x} = t$. Integrating, $\dot{x} = \frac{1}{2}t^2 + C_1$, and integrating again, $x(t) = \frac{1}{6}t^3 + C_1 t + C_2$, for arbitrary constants $C_1$, $C_2$.

Now, for $a \neq 0$ let us find a particular solution of the complete equation. It will of the form $x_p(t) = At + B$. Plugging this into the equation we find

$$a(At + B) = t \Rightarrow A = -\frac{1}{a}, \quad B = 0.$$

Hence $x_p(t) = -1/a$. To find $x_h$ we solve the characteristic equation $r^2 - a = 0$ and distinguish two cases.

(i) $a > 0$. Then $x_h(t) = C_1 e^{\sqrt{a}t} + C_2 e^{-\sqrt{a}t}$ and

$$x(t) = C_1 e^{\sqrt{a}t} + C_2 e^{-\sqrt{a}t} + \frac{1}{a}.$$

(ii) $a < 0$. Then $x_h(t) = C_1 \cos(t\sqrt{-a}) + C_2 \sin(t\sqrt{-a})$ and

$$x(t) = C_1 \cos(t\sqrt{-a}) + C_2 \sin(t\sqrt{-a}) + \frac{1}{a}.$$

(b) The general solution is $e^t(C_1 + C_2 t) + \frac{\cos t}{2}$ and the solution satisfying the initial conditions is

$$e^t\left(\frac{1}{2} + \frac{t}{2}\right) + \frac{\cos t}{2}.$$

(c) The general homogeneous solution is $x_h(t) = C_1 e^{2t} + C_2 e^t$ and a particular solution of the complete equation is $x_p(t) = (0.5t^2 - t + 1)e^{3t}$. Hence

$$x(t) = x_h(t) + x_p(t) = C_1 e^{2t} + C_2 e^t + (0.5t^2 - t + 1)e^{3t}.$$

7-3. *An equation of the form*

$$t^2 x'' + atx' + bx = 0,$$

*where $a$ and $b$ are real constants, is called an* Euler equation. *Show that the substitution of the independent variable $s = \ln t$ transforms an Euler equation into an equation with constant coefficients for the new dependent variable $y(s) = x(e^s)$. As an application, find the solution of the equation $t^2x'' - 4tx' - 6x = 0$ for $t > 0$.*

**Solution:** The equation

$$t^2 x'' + atx' + bx = 0$$

transforms into a linear one by a change of "time" $t = e^s$, or $s = \ln t$. Notice that $t'(s) = e^s = t$. Let us define a new function $y(s) = x(t) = x(e^s)$. We have, applying the chain rule

$$y'(s) = x'(t)t' = x'(t)t,$$
$$y''(s) = (x'(t)t)' = x''(t)t't + x'(t)t' = x''(t)t^2 + x'(t)t = x''(t)t^2 + y'(s).$$

Thus, we have found $tx' = y'$ and $t^2x'' = y'' - y'$. Substituting into the equation we have

$$y''(s) - y'(s) + ay'(s) + by(s) = 0,$$

which is a linear equation for $y(s)$.

The equation $t^2x'' - 4tx' - 6x = 0$ has $a = -4$ and $b = -6$, thus it transforms into

$$y'' - 5y' - 6y = 0.$$

The solution is

$$y(s) = C_1 e^{6s} + C_2 e^{-s} \Rightarrow x(t) = y(\ln t) = C_1 e^{6\ln t} + C_2 e^{-\ln t} = C_1 t^6 + C_2 t^{-1}.$$

7-4. *Suppose that a risky asset $X$ grows at an average exponential rate of $\alpha$ but it is subjected to random fluctuations of instantaneous volatility $\sigma$. Let $V(x)$ be the value of a security that collects $x\,dt$ euros continuously when the price of the stock is $X = x$. Supposing that the risk free interest rate in the economy is $r < \alpha$, it can be shown by arbitrage reasonings that the value of the stock $V(x)$ satisfies the equation the Euler equation*

$$\frac{\sigma^2}{2}x^2 V''(x) + axV'(x) - rV(x) = x.$$

*Find the general solution and pick up the economically sensible solution among these.*

**Solution:** Let us write the equation as

$$x^2 V''(x) + axV'(x) - bV(x) = cx,$$

where

$$a = \frac{2\alpha}{\sigma^2},$$
$$b = \frac{2r}{\sigma^2} > 0,$$
$$c = \frac{1}{2\sigma^2}.$$

By the problem above, the change $s = \ln x$ ($x = e^s$) transforms the equation into

(1) $$y''(s) + (a-1)y'(s) - by(s) = ce^s.$$

The general solution of the homogeneous equation is

$$C_1 e^{r_1 s} + C_2 e^{r_2 s},$$

since the roots of the characteristic equation $r^2 + (a-1)r - b = 0$ are both real and distinct,

$$r_{1,2} = \frac{1}{2}(1 - a \pm \sqrt{(a-1)^2 + 4b}\,).$$

A particular solution we guess the form $Ae^s$, so that substituting into (1) we have

$$A(1 + (a-1) - b) = c \Rightarrow A = \frac{c}{a-b} = \frac{1}{\alpha - r}.$$

We have found thus that

$$y(s) = C_1 e^{r_1 s} + C_2 e^{r_2 s} + \frac{1}{\alpha - r} e^s.$$

Turning back to the original variables

$$V(x) = y(\ln x) = C_1 x^{r_1} + C_2 x^{r_2} + \frac{1}{\alpha - r} x.$$

Finally, to select the solution that makes sense, notice that it is plausible that if $X$ doubles its value to $x$, then the value $V(2x) = 2V(x)$. Since that in general both $r_1$ and $r_2$ are $\neq 0$, we need to impose $C_1 = C_2 = 0$, so that

$$V(x) = \frac{1}{\alpha - r} x$$

should be the “correct” value.

7-5. *Let the demand and supply functions for a single commodity be given by*

$$D(t) = 42 - 4P(t) - 4\dot{P}(t) + \ddot{P}(t),$$
$$S(t) = -6 + 8P(t).$$

*We have assumed that the demand depends not only on current price, $P$, but also in expectations about the first and second variation of prices, given by $\dot{P}$ and $\ddot{P}$, respectively. Assuming that market clears at every time $t$, i.e. $D(t) = S(t)$, determine the path of $P$. Determine a linear relation between initial conditions $P(0)$ and $\dot{P}(0)$ such that the solution is bounded.*

**Solution:** In equilibrium,

$$42 - 4P(t) - 4\dot{P}(t) + \ddot{P}(t) = -6 + 8P(t)$$

thereby

$$\ddot{P}(t) - 4\dot{P}(t) - 12P(t) = -48.$$

The solution is $P(t) = C_1 e^{6t} + C_2 e^{-2t} + 4$. For having bounded solutions it is needed $C_1 = 0$. Since

$$P(0) = C_1 + C_2 + 4,$$
$$\dot{P}(0) = 6C_1 - 2C_2,$$

imposing $C_1 = 0$ we get

$$2P(0) + \dot{P}(0) = 8.$$

7-6. *An entomologist is studying two neighboring populations of red and black ants. She has estimated that the number of black ants is approximately 60,000 and that of red ants is 15,000. The ants begin fighting and our entomologist observe that at any time, the number of ants killed of one population is proportional to the number of ants alive of the other population. However, red ants are more aggressive than black ants in such a way that their effectiveness in the fight is quadruple that of black ants. The observer receives a call to her mobile phone and must leave the observation, coming back to the camp. She knows that these two species of ants fight until one of them is annihilated. She conjecture that, given that the initial population of ants is 4:1 in favor of blacks, but the effectiveness is 4:1 for reds, both populations will practically extinct at once. However, when she returns next day* *to the anthill, the situation is quite different. Could you help our hero to understand what happened by answering the following questions?*

(a) *Which is the survival species?*

(b) *How many ants of the survival species remain alive?*

(c) *Which should be the initial proportion of both populations in order that both species become extinct at once?*

*Hint: Denoting $x(t) =$ black ants at time $t$, $y(t) =$ red ants at time $t$ (both in thousands), justify why the interaction between ants can be given by*

$$\dot{x}(t) = -4ky(t),$$
$$\dot{y}(t) = -kx(t),$$

*with $k > 0$ a constant which is the fight effectiveness of black ants. This system can be converted into a second order ODE for $x(t)$ alone (or for $y(t)$). Then, solve and find the paths of $x(t)$ and $y(t)$, knowing that $x(0) = 60$ and $y(0) = 15$.*

**Solution:** The system

$$\dot{x}(t) = -4ky(t),$$
$$\dot{y}(t) = -kx(t),$$

can be analyzed in the following way. Write $dx = \dot{x}$ and $dy = \dot{y}$ and divide both equations to obtain

$$\frac{dx}{dy} = 4\frac{y}{x} \Rightarrow x\,dx - 4y\,dy = 0.$$

This is an exact equation that can be solved to get

$$\frac{x^2}{2} - 2y^2 = C.$$

Using the initial conditions we can determine $C$:

$$\frac{x^2(0)}{2} - 2y^2(0) = C \Rightarrow C = \frac{60^2}{2} - 215^2 = 1350.$$

Hence the evolution of both populations of ants is linked by

$$\frac{x^2(t)}{2} - 2y^2(t) = 1350.$$

(a) Notice that $x(t) = 0$ is impossible, but $y(t) = 0$ is possible, hence the survival species is that of black ants.

(b) Plugging $y = 0$ we find

$$x^2 = 2700 \Rightarrow x \approx 51962 \text{ black ants.}$$

(c) It is impossible that $x(s) = y(s) = 0$ at some finite $s$, but it could be $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$.

To see this, let us find the time–path of the populations by means of a second order differential equation. Deriving in $\dot{x} = -4ky$ we have

$$\ddot{x} = -4k\dot{y} = 4k^2x.$$

This equation holds whenever $y > 0$, because if $y = 0$, then $\ddot{x} = \dot{x} = 0$. The solution is

$$x(t) = C_1e^{2kt} + C_2e^{-2kt}.$$

Thus,

$$y(t) = -\frac{1}{2}C_1e^{2kt} + \frac{1}{2}C_2e^{-2kt}.$$

If we choose initial populations $x_0$, $y_0$ such that $C_1 = 0$, then the limit is 0. Thus we need

$$x_0 = C_1 + C_2,$$
$$y_0 = -\frac{C_1}{2} + \frac{C_2}{2}.$$

Plugging $C_1 = 0$ and eliminating $C_2$ we find $x_0 = 2y_0$, that is, it should be half red ants than black ants.

To complete this problem, let us find the solution with the data given above. The constants are determined with the initial conditions, $x(0) = 60$ and $\dot{x}(0) = -4ky(0) = -60k$. Therefore

$$\begin{aligned} 60 &= C_1 + C_2, \\ -60k &= 2kC_1 - 2kC_2. \end{aligned}$$

Solving, we get $C_1 = 15$ and $C_2 = 45$. Hence

$$x(t) = 15e^{2kt} + 45e^{-2kt} \qquad (y(t) > 0).$$

On the other hand $y = -\frac{1}{4k}\dot{x}$, thus

$$y(t) = 7.5e^{2kt} - 22.5e^{-2kt}.$$

Red ants become extinct at time

$$\widehat{t} = \frac{\ln 3}{k} \approx \frac{0.275}{k}.$$

The evolution of black ants is thus

$$x(t) = \begin{cases} 15e^{2kt} + 45e^{-2kt}, & \text{if } 0 \le t \le \widehat{t}; \\ 51962, & \text{if } t \ge \widehat{t}. \end{cases}$$

7-7. *Consider the functional equation*

(2)
$$g(x) + \alpha \int_0^x (x-t)f(t)\,dt = f(x), \quad \textit{for all } x \ge 0,$$

*where $g : [0,\infty) \longrightarrow \mathbb{R}$ is a given function, of class $C^2$, $\alpha \neq 0$ is a given constant and the function $f : [0,\infty) \longrightarrow \mathbb{R}$ is the unknown function, a solution of class $C^2$ to the functional equation.*

(a) *Show that if $f$ is a solution of (2), then it satisfies the second order ODE with initial values:*

$$f''(x) - \alpha f(x) = g''(x), \quad f(0) = g(0),\ f'(0) = g'(0).$$

(b) *Using (a) above, find the candidate solution $f$ in the following cases*[1]*:*
  (i) *$g(x) = ax + b$, $a, b \in \mathbb{R}$ (there are two cases to consider, $\alpha > 0$ and $\alpha < 0$.)*
  (ii) *$|\alpha| = 1$ and $g(x) = e^{ax}$ (for $\alpha = 1$ consider the cases $a = 1$, $a = -1$ and $|a| \neq 1$.)*

**Solution:** (a) If we derive the functional equation (for this we need to use Leibniz's Rule of derivation of parametric integrals), we obtain that the function $f$ must satisfy

$$g'(x) + (x-x)f(x) + \alpha \int_0^x \left(\frac{\partial}{\partial x}(x-t)f(t)\right) dt = f'(x),$$

that is,

(3)
$$g'(x) + \alpha \int_0^x f(t)\,dt = f'(x).$$

Deriving this equation, we obtain

$$g''(x) + \alpha f(x) = f''(x), \text{ or } f''(x) - \alpha f(x) = g''(x).$$

On the other hand, from the functional equation we get the initial condition

$$f(0) = g(0) + \alpha \int_0^0 (x-t)f(t)\,dt = g(0)$$

and from (3),

$$f'(0) = g'(0) + \alpha \int_0^0 f(t)\,dt = g'(0).$$

[1]It can be proved that it is indeed the only solution of class $C^2$ of (2)

(bi) Note that $g''(x) = 0$, thus the ODE satisfied by $f$ is homogeneous of constant coefficients

$$f''(x) - \alpha f(x) = 0.$$

The roots of the characteristic polynomial are real if $\alpha > 0$, $\pm\sqrt{\alpha}$, and complex if $\alpha < 0$, $\pm i\sqrt{|\alpha|}$. Thus, the general solution of the ODE is

$$C_1 e^{\sqrt{\alpha}x} + C_2 e^{-\sqrt{\alpha}x}, \quad \text{if } \alpha > 0, \tag{4}$$

$$C_1 \cos\sqrt{|\alpha|}x + C_2 \sin\sqrt{|\alpha|}x, \quad \text{if } \alpha < 0. \tag{5}$$

The initial conditions are $f(0) = g(0) = b$ and $f'(0) = g'(0) = a$, thus we have to solve the systems

$$(\text{if } \alpha > 0) \begin{cases} b = C_1 + C_2 \\ a = C_1\sqrt{\alpha} - C_2\sqrt{\alpha} \end{cases} \qquad (\text{if } \alpha < 0) \begin{cases} b = C_1 \\ a = \sqrt{|\alpha|}C_2 \end{cases}$$

to isolate the suitable solutions from the general solutions given in (4)-(5) above.

In the case $\alpha > 0$ the solution is

$$\left(\frac{b\sqrt{\alpha} + a}{2\sqrt{\alpha}}\right) e^{\sqrt{\alpha}x} + \left(\frac{b\sqrt{\alpha} - a}{2\sqrt{\alpha}}\right) e^{-\sqrt{\alpha}x}$$

and in the case $\alpha < 0$ the solution is

$$b\cos\sqrt{|\alpha|}x + \frac{a}{\sqrt{|\alpha|}} \sin\sqrt{|\alpha|}x.$$

(bii) The ODE is non-homogeneous:

$$f''(x) - \alpha f(x) = e^{ax}.$$

- When $\alpha = 1$ and $a = 1$, a particular solution is of the form: $Axe^x$. Plugging this choice into the ODE we get $A(x+2)e^x - Axe^x = e^x$, that is, $A = \frac{1}{2}$. Hence, the general solution is

  $$C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x.$$

  The initial conditions are $f(0) = g(0) = 1$ and $f'(0) = g'(0) = 1$, hence we set the system:

  $$\begin{cases} 1 = C_1 + C_2 \\ 1 = C_1 - C_2 + \frac{1}{2}. \end{cases}$$

  Solving for $C_1$ and $C_2$ gives the solution to (2)

  $$f(x) = \frac{1}{4} e^x + \frac{3}{4} e^{-x} + \frac{1}{2} x e^x.$$

- When $\alpha = 1$ and $a = -1$, a particular solution is of the form: $Axe^{-x}$. Plugging this choice into the ODE we get $A(x-2)e^{-x} - Axe^{-x} = e^{-x}$, that is, $A = -\frac{1}{3}$. Hence, the general solution is

  $$C_1 e^x + C_2 e^{-x} - \frac{1}{3} x e^{-x}.$$

  The initial conditions are $f(0) = g(0) = 1$ and $f'(0) = g'(0) = -1$, hence we set the system:

  $$\begin{cases} 1 = C_1 + C_2 \\ -1 = C_1 - C_2 - \frac{1}{3}. \end{cases}$$

  Solving for $C_1$ and $C_2$ gives the solution to (2)

  $$f(x) = \frac{1}{6} e^x + \frac{5}{6} e^{-x} - \frac{1}{3} x e^{-x}.$$

- When $\alpha = 1$ and $|a| \neq 1$, a particular solution is of the form: $Ae^{ax}$. Plugging this choice into the ODE we get $Aa^2 e^{ax} - Ae^{ax} = e^{ax}$, that is, $A = \frac{1}{a^2-1}$. Hence, the general solution is

  $$C_1 e^x + C_2 e^{-x} + \frac{1}{a^2 - 1} e^x.$$

The initial conditions are $f(0) = g(0) = 1$ and $f'(0) = g'(0) = a$, hence we set the system:

$$\begin{cases} 1 = C_1 + C_2 + \frac{1}{a^2-1} \\ a = C_1 - C_2 + \frac{a}{a^2-1}. \end{cases}$$

Solving for $C_1$ and $C_2$ gives the solution to (2) (complete the details!).

- When $\alpha = -1$, a particular solution is $Ae^{ax}$ (for all $a$) as in the previous case and we obtain now $A = \frac{1}{a^2+1}$. Hence, the general solution is

$$C_1 \cos\sqrt{|\alpha|}x + C_1 \sin\sqrt{|\alpha|}x + \frac{1}{a^2+1}e^{ax}.$$

The initial conditions are again $f(0) = g(0) = 1$ and $f'(0) = g'(0) = a$, hence we set the system:

$$\begin{cases} 1 = C_1 + \frac{1}{a^2+1} \\ a = -C_2 + \frac{a}{a^2+1}. \end{cases}$$

Solving for $C_1$ and $C_2$ gives the solution to (2) (complete the details!).

7-8. *Let $u(x)$ be the utility obtained from wealth $x$. Function $u$ is strictly increasing ($u' > 0$) and concave ($u'' < 0$). The Arrow–Pratt measure of absolute risk aversion, $r(x)$, depends on wealth level, and it is the defined as the proportional change in the marginal utility, $u'(x)$*

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

(a) *Find all functions with Constant Absolute Risk Aversion (CARA utility functions for short), i.e. with $r(x) \equiv a > 0$.*

(b) *Find all functions with Arrow–Pratt index inversely proportional to wealth, i.e. $r(x) = a/x$, $a > 0$. These functions are called of Constant Relative Risk Aversion (CRRA utility functions).* Hint*: Transform the second order equation* with non–constant coefficients *to a first order equation for $v(x) = u'(x)$. Solve the first order equation you obtain for $v$ and then find $u$ by direct integration. Possibly you need to distinguish two cases, $a \neq 1$ and $a = 1$.*

**Solution:**

(a) The ODE for $u$ is $u'' + au' = 0$, which general solution is $u(x) = C_1 e^{-ax} + C_2$.

(b) The ODE for $u$ is $u'' + \frac{a}{x}u' = 0$, that can be converted into a firs order ODE for $v = u'$,

$$v' + \frac{a}{x}v = 0.$$

This gives $\left(ve^{\int \frac{a}{x}\,dx}\right)' = 0$, that is, $ve^{\ln x^a} = C_1$ and solving for $v$

$$v(x) = C_1 x^{-a}.$$

Integrating we get $u(x) = \int v'$, that is, $u(x) = C_1\frac{x^{1-a}}{1-a} + C_2$, if $a \neq 1$, and $u(x) = C_1 \ln x + C_2$, if $a = 1$.