ADVANCED MATHEMATICS FOR ECONOMICS – 2016/2017

Sheet 7. Differential Equations (3)

7-1. Answer the following questions:
  (a) Form the homogeneous linear ODEs from their characteristic equation.
    (i) $r^2 - 3r + 5 = 0$;
    (ii) $r(r+2) = 0$.
  (b) Form the homogeneous linear ODEs from the roots of their characteristic equation.
    (i) $r_1 = 1$, $r_2 = 4$;
    (ii) $r_1 = 3 - 4i$, $r_2 = 3 + 4i$.
  (c) Form the homogeneous linear ODEs from their general solution.
    (i) $C_1 e^t + C_2 e^{-2t}$;
    (ii) $C_1 e^{-2t} + C_2 t e^{-2t}$;
    (iii) $e^{-t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

7-2. Find the solution of the following equations.
  (a) $x'' - ax = t$, where $a \in \mathbb{R}$,
  (b) $x'' - 2x + x = \sin t$, $x(0) = \dot{x}(0) = 1$.
  (c) $x'' - 3x' + 2x = (t^2 + t)e^{3t}$,

7-3. An equation of the form
$$t^2 x'' + atx' + bx = 0,$$
where a and b are real constants, is called an *Euler equation.* Show that the substitution of the independent variable $s = \ln t$ transforms an Euler equation into an equation with constant coefficients for the new dependent variable $y(s) = x(e^s)$. As an application, find the solution of the equation $t^2 x'' - 4tx' - 6x = 0$ for $t > 0$.

7-4. Suppose that a risky asset X grows at an average exponential rate of α but it is subjected to random fluctuations of instantaneous volatility σ. Let $V(x)$ be the value of a security that collects $x\,dt$ euros continuously when the price of the stock is $X = x$. Supposing that the risk free interest rate in the economy is $r < \alpha$, it can be shown by arbitrage reasonings that the value of the stock $V(x)$ satisfies the equation the Euler equation
$$\frac{\sigma^2}{2} x^2 V''(x) + axV'(x) - rV(x) = x.$$
Find the general solution and pick up the economically sensible solution among these.

7-5. Let the demand and supply functions for a single commodity be given by
$$D(t) = 42 - 4P(t) - 4\dot{P}(t) + \ddot{P}(t),$$
$$S(t) = -6 + 8P(t).$$
We have assumed that the demand depends not only on current price, P, but also in expectations about the first and second variation of prices, given by $\dot{P}$ and $\ddot{P}$, respectively. Assuming that market clears at every time t, i.e. $D(t) = S(t)$, determine the path of P. Determine a linear relation between initial conditions $P(0)$ and $\dot{P}(0)$ such that the solution is bounded.

7-6. An entomologist is studying two neighboring populations of red and black ants. She has estimated that the number of black ants is approximately $60,000$ and that of red ants is $15,000$. The ants begin fighting and our entomologist observe that at any time, the number of ants killed of one population is proportional to the number of ants alive of the other population. However, red ants are more aggressive than black ants in such a way that their effectiveness in the fight is quadruple that of black ants. The observer receives a call to her mobile phone and must leave the observation, coming back to the camp. She knows that these two species of ants fight until one of them is annihilated. She conjecture that, given that the initial population of ants is 4:1 in favor of blacks, but the effectiveness is 4:1 for reds, both populations will practically extinct at once. However, when she returns next day to the anthill, the situation is quite different. Could you help our hero to understand what happened by answering the following questions?

(a) Which is the survival species?

(b) How many ants of the survival species remain alive?

(c) Which should be the initial proportion of both populations in order that both species become extinct at once?

Hint: Denoting $x(t)$ = black ants at time t, $y(t)$ = red ants at time t (both in thousands), justify why the interaction between ants can be given by

$$\dot{x}(t) = -4ky(t),$$
$$\dot{y}(t) = -kx(t),$$

with $k > 0$ a constant which is the fight effectiveness of black ants. This system can be converted into a second order ODE for $x(t)$ alone (or for $y(t)$). Then, solve and find the paths of $x(t)$ and $y(t)$, knowing that $x(0) = 60$ and $y(0) = 15$.