ADVANCED MATHEMATICS FOR ECONOMICS – 2014/2015

Sheet 6. Differential Equations (2)

6-1. Solve the following first order ODEs.

(a) $\dot{x} = \frac{t^3}{x^3}$.
(b) $\dot{x} = \frac{x^3}{t^3}$.
(c) $y' = \frac{\sqrt{x+1}}{y^2}$, with $y(0) = \frac{5}{3}$.
(d) $(t + 3x)\,dt + (t - 20)\,dx = 0$.
(e) $(2xy - \cos x)\,dx + (x^2 - 1)\,dy = 0$, with $y(0) = 0$.
(f) $\dot{x} + 2tx = \cos te^{-t^2}$, with $x(0) = 0$.
(g) $\dot{x} + \frac{x}{t} = e^{-t^2}$.

6-2. The equation

$$\dot{x} + a(t)x = b(t)x^n$$

is a *Bernoulli equation*. It is a linear equation for $n = 0$ or $n = 1$, but it is not linear for $n \neq 0, 1$. Suppose that this is the case.

(a) Prove that the change of variable $y = x^{1-n}$ transforms the equation into a linear equation for $y(t)$.
(b) Solve $\dot{x} + 2x = x^3$, $x(0) = 2$.

6-3. Draw the phase diagrams of each of the following equations, find the equilibrium points and study their stability.

(a) $\dot{x} = g_1(x) = (x + 1)(x - 1)^2(x - 2)$.
(b) $\dot{x} = g_2(x) = (x + 1)(x - 1)(x - 2)$

6-4. Suppose that the population y of a certain species of fish in a given area of the ocean is described by the logistic equation

$$\dot{y} = r\left(1 - \frac{y}{K}\right)y.$$

The resource is used for food. Suppose that the rate at which fish are caught, $E(y)$, is proportional to the population y. Thus, we assume that $E(y) = Ey$, with E a positive constant. Then the logistic equation is replaced by

$$\dot{y} = r\left(1 - \frac{y}{K}\right)y - Ey.$$

This equation is known as *Schaefer model*

(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$, $y_2 > 0$;
(b) Show that y_1 is unstable and y_2 is asymptotically stable.
(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is defined as Ey_2. Find Y as a function of the effort E and graph the function (it is known as the yield–effort curve).
(d) Determine E so as to maximize Y and thereby find the *maximum sustainable yield* Y_m.

6-5. Consider the following growth model of Haavelmo. The output in the economy is given by a Cobb–Douglas production function with constant returns to scale

$$Y = F(K, L) = AK^aL^{1-a}, \qquad 0 < a < 1,$$

where K is capital stock and L is level of employment; the constant $A > 0$, but we will take $A = 1$. It is supposed that the rate of growth of employment is not constant, but given by

(1) $$\frac{\dot{L}}{L} = \alpha - \beta \frac{L}{Y} = \alpha - \beta \frac{1}{Y/L}, \qquad \alpha, \beta > 0.$$

thus, the rate of growth of employment is an increasing function of per capita income (output), Y/L. Notice that the capital stock, K, is considered *constant*. Plugging Y into (1) we get the ODE

$$\dot{L} = \alpha L - \beta \frac{L^{1+a}}{K^a}, \qquad L(0) = L_0 > 0.$$

(a) Find the equilibrium points (if any) and draw the phase diagram of the ODE.
(b) Study the asymptotic behavior of the solution.

6-6. Five college students with the flu return after Christmas Holidays to an isolated campus of 2500 students. If the rate at which this virus spreads is proportional to the number of infected students y and to the number not infected $2500 - y$, solve the initial value problem

$$\dot{y} = ky(2500 - y), \qquad y(0) = 5$$

to find the number of infected students after t days if 25 students have the virus after one day. How many students have the flu after five days? Determine the number of days required for half the campus to be infected.