TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS

Sheet 5. Differential Equations (1)

Solutions

5-1. *Is the equation $\dot{x}(t) = x(t^2)$ an ordinary differential equation?*

Solution: No, it is a functional equation.

5-2. *Check that $x(t) = \pm\sqrt{\ln(C(t^2+1))}$, where C is a positive constant, is solution of the ODE*

$$\dot{x}(t) = \frac{t}{x(t)(t^2+1)}.$$

Solution: It is only a matter of computing.

5-3. *Write the second order ODE*

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = c(t)$$

as a first order system.

Solution: Let $y = \dot{x}$. Then

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -a(t)y - b(t)x + c(t). \end{aligned}$$

5-4. *It is snowing with regularity. At 12 am, a snow plow began to remove snow. The machine took 2 km. in the first hour, and only 1 km. in the second hour. Knowing that the snow plow remove a constant amount of snow per unit of time, When did it start snowing?*

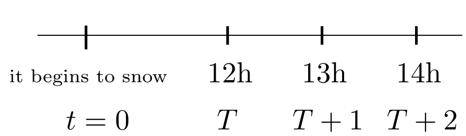
Solution: Let $x(t)$ the function given the displacement (in km.) of the snow plow. As we know from elementary Physics, the velocity is the derivative of the displacement, $v(t) = \dot{x}(t)$. The conditions of the problem say that the velocity of the snow plow is inversely proportional to the height of snow at time t, $h(t)$. But the height is proportional to the time elapsed, $h(t) = k_1 t$, with $k_1 > 0$. Hence,

$$\dot{x}(t) = v(t) = \frac{k_2}{h(t)} = \frac{k_2}{k_1 t}, \qquad k_2 > 0.$$

This is a very simple ODE, with solution

$$x(t) = k \ln t + C, \qquad k = \frac{k_2}{k_1}.$$

Let T be the period of time (in hours) that it has been snowing before 12 am and observe the following picture.



We have the following data:

$$t = T \Rightarrow x(T) = 0 = k\ln T + C,$$
$$t = T+1 \Rightarrow x(T+1) = 2 = k\ln(T+1) + C,$$
$$t = T+2 \Rightarrow x(T+2) = 3 = k\ln(T+2) + C.$$

Plugging $C = -k\ln T$ into the two remainder equations we have

$$2 = k(\ln(T+1) - \ln(T)) = k\ln\left(\frac{T+1}{T}\right),$$
$$3 = k(\ln(T+2) - \ln(T)) = k\ln\left(\frac{T+2}{T}\right),$$

Dividing both equations to eliminate k we get

$$2\ln\left(\frac{T+2}{T}\right) = 3\ln\left(\frac{T+1}{T}\right),$$

or

$$\left(\frac{T+2}{T}\right)^2 = \left(\frac{T+1}{T}\right)^3.$$

We find an equation for T:

$$T(T^2+4T+4) = T^3+3T^2+3T+1$$

or

$$T^2+T-1=0.$$

Solving gives $T = 0{,}618\ldots$ hours, that is, 37 minutes and 5 seconds approximately, thus it began to snow at $11:22:55$ am.

5-5. *Find the solution of the following problems:*

(a) $\dot{x} = \dfrac{e^t}{x(1+e^t)}$.

(b) $\dot{x} = e^{t-x}$, $x(0) = 1$.

Solution:

Both equations are separable.

(a)

$$\dot{x}x = \frac{e^t}{(1+e^t)}$$
$$\int x\,dx = \int \frac{e^t\,dt}{(1+e^t)}$$
$$\frac{x^2}{2} = \ln(1+e^t) + C.$$

(b)

$$\dot{x}e^x = e^t$$
$$\int e^x\,dx = \int e^t\,dt$$
$$e^x = e^t + C$$
$$x(t) = \ln(e^t + C).$$

Plugging $x(0) = 1$ into the above expression we have

$$1 = \ln(1+C).$$

Hence, $C = e-1$ and thus $x(t) = \ln(e^t + e - 1)$.

5-6. *Show that a separable equation is exact.*

Solution: The ODE $P\,dt + Q\,dx = 0$ is exact if $\partial P/\partial x = \partial Q/\partial t$. For a separable equation

$$\dot{x} = g(t)h(x)$$

we can write

$$-g(t)\,dt + \frac{dx}{h(x)} = 0.$$

Since

$$\frac{\partial}{\partial x} g(t) = 0 = \frac{\partial}{\partial t}\frac{1}{h(x)},$$

we are done.

5-7. *Check whether the following ODEs are exact and solve them.*
(a) $(\alpha t + \beta x)\,dt + (\beta t + \gamma x)\,dx = 0.$
(b) $(-2tx^3 + t\ln t)\,dt - (3t^2x^2)\,dx = 0.$

Solution: Both are exact.
(a) We know

$$V(t,x) = \int P\,dt = \int \alpha t + \beta x\,dt = \alpha\frac{t^2}{2} + \beta t x + \psi(x).$$

Hence

$$V_x = \beta t + \psi'(x).$$

On the other hand, $V_x = Q = \beta t + \gamma x$, thus

$$\psi'(x) = \gamma x \Rightarrow \psi(x) = \gamma\frac{x^2}{2}.$$

Finally, the solution is given by $V(t, x(t)) = C$, hence

$$\alpha\frac{t^2}{2} + \beta t x(t) + \gamma\frac{x^2}{2} = C$$

is the general solution.
(b)

$$V(t,x) = \int Q\,dx = -3\int t^2x^2\,dx = -t^2x^3 + \psi(t)$$

thus,

$$V_t = -2tx^3 + \psi'(t).$$

On the other hand, $V_t = P = -2tx^3 + t\ln t$, hence

$$\psi'(t) = t\ln t.$$

Using integration by parts we get $\psi(t) = \int t\ln t\,dt = \frac{t^2}{2}\ln t - \frac{t^2}{4}$.
Parts: $\int u\,dv = uv - \int v\,du$. Take here $u = \ln t$ and $dv = t\,dt$. Then, $du = dt/t$ and $v = t^2/2$ and so on.

5-8. *Consider the following supply and demand functions: $Q_s(P) = P - 6$, $Q_d(P) = 15 - 2P$. The prices depends on time, $P(t)$, which derivative satisfies*

$$\dot{P} = 2(Q_d(P) - Q_s(P)).$$

Calculate $P(t)$, the equilibrium price and the equilibrium quantity and study if the price converges to the equilibrium price in the long run.