TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS – 2020/2021

Sheet 3. Difference Equations (2)

Solutions

3-1. *Find the solutions of the equations (a)* $x_{t+2} - \frac{1}{4}x_t = \sin\frac{\pi}{2}t$, $x_0 = 1/2$, $x_1 = 0$; *(b)* $x_{t+2} - x_{t+1} + x_t = e^{-t} + 1$.

Solution:

(a) The characteristic equation is $r^2 - \frac{1}{4} = 0$, whose roots are $r_{1,2} = \pm\frac{1}{2}$. The general solution of the homogeneous equation is $A2^{-t} + B(-2)^{-t}$, $A, B \in \mathbb{R}$. A particular solution is of the form

$$x_t^* = C\sin\pi t/2 + D\cos\pi t/2, \qquad C, D \in \mathbb{R};$$

To identify these constants we substitute x_t^* into the equation to get

(1) $$C\sin\frac{\pi}{2}(t+2) + D\cos\frac{\pi}{2}(t+2) - \frac{1}{4}\left(C\sin\frac{\pi}{2}t + D\cos\frac{\pi}{2}t\right) = \sin\frac{\pi}{2}t.$$

Recalling that

$$\sin\frac{\pi}{2}(t+2) = \sin\frac{\pi}{2}t\cos\pi + \cos\frac{\pi}{2}t\sin\pi = -\sin\frac{\pi}{2}t$$
$$\cos\frac{\pi}{2}(t+2) = \cos\frac{\pi}{2}t\cos\pi - \sin\frac{\pi}{2}t\sin\pi = -\cos\frac{\pi}{2}t$$

and collecting terms we have from (**??**)

$$-C - \frac{1}{4}C = 1,$$
$$-D - \frac{1}{4}D = 0.$$

Thus, $C = -4/5$ and $D = 0$. The general solution is

(2) $$x_t = A2^{-t} + B(-2)^{-t} - \frac{4}{5}\sin\pi t/2.$$

To find the solution satisfying the initial conditions $x_0 = 1/2$ and $x_1 = 0$ we need to determine the constants A and B in the expression (**??**). We obtain the system

$$\frac{1}{2} = x_0 = A + B,$$
$$0 = x_1 = \frac{A}{2} - \frac{B}{2} - \frac{4}{5}$$

Solving we get $A = 21/20$ and $B = -11/20$. Therefore the solution is

$$x_t = \frac{21}{20}2^{-t} + \frac{11}{20}(-2)^{-t} - \frac{4}{5}\sin\pi t/2.$$

(b) The characteristic equation is $r^2 - r + 1 = 0$, whose roots are $r_{1,2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The modulus is $\rho = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$. The general solution of the homogeneous equation is

$$A\cos\theta t + B\sin\theta t,$$

where $\tan\theta = \sqrt{3}$, hence $\theta = \pi/3$. A particular solution is of the form $x_t^* = Ce^{-t} + D$. Plugging this expression into the equation we have

$$Ce^{-t-2} + D - Ce^{-t-1} - D + Ce^{-t} + D = e^{-t} + 1.$$

Thus,

$$Ce^{-2} - Ce^{-1} + C = 1,$$
$$D = 1.$$

We get $C = (e^{-2} - e^{-1} + 1)^{-1}$ and $D = 1$. The general solution is thus

$$x_t = A\cos\frac{\pi}{3}t + B\sin\frac{\pi}{3}t + \frac{e^{-t}}{e^{-2} - e^{-1} + 1} + 1.$$

3-2. *Investigate the stability of the following equations: (a) $x_{t+1} - \frac{1}{4}x_t = b_t$, (b) $x_{t+2} - x_{t+1} + x_t = c_t$, where $\{b_t\}$ and $\{c_t\}$ are given sequences.*

Solution: Recall that the stability of a linear equation of second order depends exclusively on the behavior of the general solution of the homogeneous equation and this in turn depends on the roots of the characteristic equation being strictly less than 1 in module.

(a) The characteristic equation is $r^2 - \frac{1}{4} = 0$, whose roots are $r_{1,2} = \pm\frac{1}{2}$. Both roots have moduli strictly less than 1, and thus the homogeneous equation converges to zero. Hence, any solution converges to the same particular solution independently of the initial conditions x_0, x_1.

(b) The characteristic equation is $r^2 - r + 1 = 0$, whose roots are $r_{1,2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The modulus is $\rho = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$, hence the equation is not g.a.s. It is stable, because the general solution of the homogeneous equation is

$$A\cos\theta t + B\sin\theta t,$$

which oscillates around 0 with uniform oscillations. Here, $\tan\theta = \sqrt{3}$, hence $\theta = \pi/3$. Thus, any solution of the complete equation oscillates around the same particular solution.

3-3. *Solve the Fibonacci equation $x_{t+2} = x_{t+1} + x_t$, $x_0 = x_1 = 1$ and check that*

$$\lim_{t\to\infty} \frac{x_{t+1}}{x_t} = \frac{1+\sqrt{5}}{2} \equiv \varphi, \quad \textit{the golden section.}$$

Solution: The characteristic equation is $r^2 - r - 1 = 0$, with roots $r_{1,2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Therefore, the general solution is

$$x_t = Ar_1^t + Br_2^t, \qquad A, B \in \mathbb{R}.$$

Imposing the initial values we get

$$\begin{aligned} A + B &= 1, \\ Ar_1 + Br_2 &= 1, \end{aligned}$$

that gives $A = r_1\sqrt{5}/5$ and $B = -r_2\sqrt{5}/5$. Hence, the solution is

$$x_t = \frac{\sqrt{5}}{5}(r_1^{t+1} - r_2^{t+1}), \qquad t = 0, 1, \ldots$$

and the limit

$$\lim_{t\to\infty} \frac{x_{t+1}}{x_t} = \lim_{t\to\infty} \frac{r_1^{t+2} - r_2^{t+2}}{r_1^{t+1} - r_2^{t+1}} = \lim_{t\to\infty} \frac{r_1 - r_2\left(\frac{r_2}{r_1}\right)^{t+1}}{1 - \left(\frac{r_2}{r_1}\right)^{t+1}} = r_1 = \frac{1+\sqrt{5}}{2},$$

since $|r_2| < r_1$.

3-4. *Consider the equation obtained in the multiplier–accelerator model of growth studied in the class notes,*

$$Y_{t+2} - a(1+c)Y_{t+1} + acY_t = b,$$

with $a > 0$, $c > 0$ and $a \neq 1$.

(a) *Find a particular solution of this equation;*

(b) *Discuss whether the solutions of the characteristic equation are real or complex.*

(c) *Find the general solution in each of the following cases.*

   (i) $a = 4$, $c = 1$;

   (ii) $a = \frac{3}{4}$, $c = 3$;

   (iii) $a = 0.5$, $c = 1$.

Solution:

(a) A particular solution is given by $Y_t^* = A$, with $A \in \mathbb{R}$. A is determined by

$$A - a(1+c)A + acA = b,$$

hence $A = b/(1-a)$.

(b) From $r^2 - a(1+c)r + ac = 0$ we get
$$r_{1,2} = \frac{1}{2}a(1+c) \pm \frac{1}{2}\sqrt{a^2(1+c)^2 - 4ac}.$$
If $a^2(1+c)^2 - 4ac > 0$ there are two distinct real roots, if it is equal zero, a double real root, and if it is < 0, two distinct complex root. In this case they are given by
$$r_{1,2} = \frac{1}{2}a(1+c) \pm i\frac{1}{2}\sqrt{|a^2(1+c)^2 - 4ac|}$$
where $i = \sqrt{-1}$ is the imaginary unit.

(c) In what follows, A and B are arbitrary constants.

(i) $r_{1,2} = 4 \pm 2\sqrt{3}$, thus
$$Y_t = A(4+2\sqrt{3})^t + B(4-2\sqrt{3})^t - \frac{b}{3}.$$
(ii) $r_{1,2} = \frac{3}{2}$, thus
$$Y_t = A\left(\frac{3}{2}\right)^t + Bt\left(\frac{3}{2}\right)^t + 4b.$$
(iii) $r_{1,2} = 0.5 \pm i\,0.5$, thus $\rho = \sqrt{2}/2$ and $\tan\theta = 1$ implies $\theta = \pi/4$. In consequence
$$Y_t = \left(\frac{\sqrt{2}}{2}\right)^t \left(A\cos\frac{\pi t}{4} + B\sin\frac{\pi t}{4}\right) + 2b.$$
Notice that the solution oscillates around the particular solution Y_t^* found in part (a) and that the oscillations are damped. In fact $\lim_{\to\infty} Y_t = 2b$ independently of the constants A, B we choose (therefore, independently of the initial values x_0, x_1) and thus the equation is g.a.s. for $a = 0.5$ and $c = 1$.

3-5. *Let C_t denotes consumption, K_t capital stock, Y_t net national product. We suppose that these variables are related as*
$$\begin{aligned} C_t &= cY_{t-1}, \\ K_t &= \sigma Y_{t-1}, \\ Y_t &= C_t + K_t - K_{t-1}, \end{aligned}$$
where c and σ are positive constants.

(a) *Give an economic interpretation of the equations.*

(b) *Derive a second order difference equation for Y_t.*

(c) *Find necessary and sufficient conditions for the solution of the equation in (b) to have explosive oscillations.*

Solution:

(a) The two first equations state that consumption and capital are proportional to the net national product in the previous period. The third equation state that the net national product is divided between consumption and net investment, $K_t - K_{t-1}$.

(b) First replace t by $t+2$ in the third equation to get $Y_{t+2} = C_{t+2} + K_{t+2} - K_{t+1}$. Since $C_{t+2} = cY_{t+1}$, $K_{t+2} = \sigma Y_{t+1}$, and $K_{t+1} = \sigma Y_t$ we obtain
$$Y_{t+2} - (c+\sigma)Y_{t+1} + \sigma Y_t = 0.$$
(c) The characteristic equation is $r^2 - (c+\sigma)r + \sigma = 0$. As we are looking for explosive oscillations we need complex solutions of the equation, with moduli bigger than 1. The discriminant, $(c+\sigma)^2 - 4\sigma$, is negative when
$$(c+\sigma)^2 < 4\sigma.$$
The module ρ of the solutions is
$$\rho = \sqrt{\sigma}.$$
It is bigger than 1 iff $\sigma > 1$.