Topic 3: Differential Equations

1. Introduction. Definitions and classifications of ODEs

Most often decision agents take optimal actions sequentially and economic variables evolve along time. Thus it is important to understand the tools of analysis and modeling of dynamical systems. We are looking at functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ or vector functions $\mathbf{x} : \mathbb{R} \longrightarrow \mathbb{R}^n$ described by equations of the form

$$\frac{d}{dt}x(t) = f(t, x(t)),$$

possibly with an initial condition $x(t_0) = x_0$.

Objectives:

(1) To find $x(t)$ in closed form or, if this is not possible,
(2) to study qualitative properties of $x(t)$ (e.g. stability).
(3) To apply the above to economic modeling.

Notation:

- x is the independent or unknown variable and t the dependent variable; most often the variable t is omitted;
- $\frac{d}{dt}x(t) \equiv \frac{dx}{dt}, x'(t)$, x', $\dot{x}(t)$, $\dot{x}$, $x^{(1)}(t)$, $x^{(1)}$.
- Higher order derivatives $\frac{d^k}{dt^k}x(t) \equiv x^{(k)}(t)$. Special case $k = 2$, x'', $\ddot{x}$, $x^{(2)}$.
- Other variables are possible, e.g. $\frac{d}{dx}y(x)$, $y'(x)$.

Definition 1.1. A one dimensional ordinary differential equation (ODE) of order k is a relation of the form

$$x^{(k)}(t) = f(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t)). \tag{1.1}$$

Note that k is the highest derivative appearing in the equation.

Definition 1.2. A first order system of ordinary differential equations is a relation of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \tag{1.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{f} = (f_1, \ldots, f_n)^T$, $x_i : \mathbb{R} \longrightarrow \mathbb{R}$, $f_i : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $i = 1, \ldots, n$.

It is always possible to transform a kth order ODE into a first order system. Let us see how. Suppose we have the kth order ODE

$$x^{(k)}(t) = f(t, x(t), x^{(1)}(t), \ldots, x^{(k-1)}(t))$$

and consider new functions defined by setting $x_1 = x, x_2 = x', \ldots, x_k = x^{k-1}$. Then the ODE appears as the following first order system of differential equations:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ &\vdots \\ \dot{x}_k &= f(t, x_1, x_2, \ldots, x_k). \end{aligned}$$

Note that in this case $\mathbf{f}(t, \mathbf{x}) = (x_2, x_3, \ldots, x_k, f(t, \mathbf{x}))$. As an example, consider the second order ODE

$$\ddot{x} = \dot{x}^2 - 2x - \cos t.$$

Let the new variable $y = \dot{x}$, Then the first order system equivalent to the original scalar equation is

$$\dot{x} = y,$$

$$\dot{y} = y^2 - 2x - \cos t.$$

Definition 1.3. A solution of the first order system (1.2) on an interval $I \subseteq \mathbb{R}$ is a differentiable function $\mathbf{x} : I \longrightarrow \mathbb{R}^n$ such that $\mathbf{x}(t) \in D$ for all $t \in I$ and $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t))$ for all $t \in I$.

Definition 1.4. An initial value problem or Cauchy problem for a first order system consists of (1.2) together with a condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $(t_0, \mathbf{x}_0) \in D$.

Definition 1.5. An initial value problem or Cauchy problem for a kth order ODE consists of (1.1) together with the conditions

$$x(t_0) = x_0, \quad x'(t_0) = x_1, \ldots, x^{(k-1)}(t_0) = x_{k-1},$$

where $(t_0, x_0, x_1, \ldots, x_k) \in D$.

Thus,

$$\dot{x} = t \sin x, \quad x(0) = \pi$$

is a Cauchy problem, as well as

$$\ddot{x} = e^{\dot{x}} - tx, \quad x(1) = 2, \quad \dot{x}(1) = -1.$$

Under suitable conditions, a Cauchy admits a unique solution.

Definition 1.6.

- The ODE (1.2) is linear if for fixed t, the map $\mathbf{x} \longrightarrow \mathbf{f}(t, \mathbf{x})$ is linear.
- The ODE (1.2) is autonomous if $\mathbf{f}$ is independent of t.

Throughout these lecture notes it is assumed the continuity of the functions f and $\mathbf{f}$.

2. Elementary integration methods of first order ODEs

Let us look at some particular cases where the scalar first order ODE

$$\dot{x}(t) = f(t, x(t)), \quad t \in I \subseteq \mathbb{R}, \quad I \text{an interval finite or infinite,} \tag{2.1}$$

can be explicitly solved.

The simplest case is when f is independent of the solution itself, x. That is

$$\dot{x}(t) = f(t), \quad t \in I = [a, b] \subseteq \mathbb{R}.$$

Finding x leads to the *integration problem.*

The Fundamental Theorem of Calculus establishes that

$$x(t) = C + \int_a^t f(t)\, dt,$$

with C a constant. If we want the solution passing through (a, x_0), then $C = x_0$. Note that even in this simple case the solution found can have little practical value, and the study of the qualitative behavior can be more illuminating. Resorting to numerical approximations of the solution is another interesting possibility.

2.1. Separable equations.

Definition 2.1. A first order ODE is separable if $f(t, x) = g(t)h(x)$, that is

$$\dot{x}(t) = g(t)h(x(t)).$$

Method of solution: Denoting $H(x)$ and $G(t)$ some antiderivatives of $1/h(x)$ and $g(t)$ respectively, observe the following steps (notice that $H' = 1/h$ and $G' = g$):

(i) Separation of variables: $\dfrac{\dot{x}}{h(x)} = g(t)$,

(ii) Chain Rule: $\dfrac{d}{dt}H(x(t)) = \dfrac{d}{dt}G(t)$,

(iii) Integration with respect to t: $H(x(t)) = G(t) + C$.

The expression obtained defines x implicitly. It is possible to prove that if $h(x_0) \neq 0$, then the solution defined by the implicit expression satisfying $x(t_0) = x_0$ is unique in a neighborhood of x_0. The constant C can be determined if an initial condition is fixed.

Example 2.2. To find the solution of the separable ODE $\dot{x} = tx^2$, we start with

$$\frac{\dot{x}}{x^2} = t \Rightarrow \frac{dx}{x^2} = t\, dt \Rightarrow \int \frac{dx}{x^2} = \int t\, dt.$$

Integrating, we find

$$-x^{-1} = \frac{t^2}{2} + C.$$

Solving for x we get

$$x(t) = -\frac{1}{\frac{t^2}{2} + C}.$$

Suppose that we want the solution passing through $(0,1)$; then

$$1 = x(0) = -\frac{1}{C} \Rightarrow C = -1,$$

thus

$$x(t) = -\frac{1}{\frac{t^2}{2} - 1}.$$

The solution exists only in the interval $(-\sqrt{2}, \sqrt{2})$.

2.2. **Exact equations. Integrating factors.** Suppose we have a first order ODE of the form in the form

$$\dot{x}(t) = -\frac{P(t, x(t))}{Q(t, x(t))}, \tag{2.2}$$

for some functions P, Q, such that $Q(t,x) \neq 0$ for every point (t,x) in some set D.

This is equivalent to $Q(t,x)\dot{x} = -P(t,x)$, and interpreting $\dfrac{dx}{dt}$ as a quotient (this has no sense, of course, but it is useful and it works in this case) we can rewrite the ODE as

$$P(t,x)\,dt + Q(t,x)\,dx = 0. \tag{2.3}$$

Consider now a function V of variables (t,x), of class C^2 (the second order partial derivatives exist and are continuous). The differential of V is

$$dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial x}\,dx.$$

Suppose that it is possible to find a function V such that

$$\frac{\partial V}{\partial t} = P. \tag{2.4}$$

$$\frac{\partial V}{\partial x} = Q. \tag{2.5}$$

Then, the differential of V

$$dV = \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial x}\,dx = P\,dt + Q\,dx = 0$$

along the solutions of the ODE. This means that V is constant. Thus we get that the solutions of the ODE are given by the implicit equation:

$$V(t, x(t)) = C.$$

This important observation motivates the following definition.

Definition 2.3. The first order ODE (2.2) (or (2.3)) is *exact* in a neighborhood D of the point (t_0, x_0) if $Q(t_0, x_0) \neq 0$ and there exists a function V of class C^2 on D satisfying (2.4) and (2.5).

When is there a function V satisfying (2.4) and (2.5)? It both conditions were true, then

$$\frac{\partial^2 V}{\partial x \partial t} = \frac{\partial P}{\partial x}. \tag{2.6}$$

$$\frac{\partial^2 V}{\partial t \partial x} = \frac{\partial Q}{\partial t}. \tag{2.7}$$

Since the order of derivation does not matter for a C^2 function,

$$\frac{\partial^2 V}{\partial x \partial t} = \frac{\partial^2 V}{\partial t \partial x},$$

we get the necessary (and sufficient) condition

$$\frac{\partial P}{\partial x}(t, x) = \frac{\partial Q}{\partial t}(t, x).$$

Theorem 2.4. *Assume that P and Q are C^1 in a neighborhood D of the point (t_0, x_0). The necessary and sufficient condition for (2.2) (or (2.3)) to be exact in D is*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} \tag{2.8}$$

in D.

Example 2.5. The equation $(2t - x^2)\, dt + 2tx\, dx = 0$ is not exact, since

$$\frac{\partial P}{\partial x} = -2x \neq 2x = \frac{\partial Q}{\partial t}.$$

However, the equation

$$(2t - x^2)\, dt - 2tx\, dx = 0$$

is exact. Let us solve. Once we determine function V, the problem is finished. To find V, we begin with (2.4)

$$\frac{\partial V}{\partial t} = P(t, x) = 2t - x^2.$$

Integrating with respect to t we get

$$V(t, x) = \int (2t - x^2)\, dt = t^2 - tx^2 + \psi(x), \tag{2.9}$$

where ψ is a function of x that we must determine using the other condition (2.5), that is,

$$\frac{\partial V}{\partial x} = Q(t, x) = -2tx.$$

Deriving in (2.9) with respect to x we get

$$\frac{\partial V}{\partial x} = -2tx + \psi'(x)$$

and equating both expressions above

$$\psi'(x) = 0.$$

We choose $\psi = 0$. Hence, $V(t,x) = t^2 - tx^2$ and since the solution satisfies $V(t, x(t)) = C$, we have

$$t^2 - tx^2(t) = C \Rightarrow x(t) = \pm\sqrt{t - \frac{C}{t}} \quad (t \neq 0).$$

If the equation

$$P(t,x)\,dt + Q(t,x)\,dx = 0$$

is not already exact, we could multiply the equation by a non null function $\mu(t,x)$ such that the equation

$$\mu(t,x)P(t,x)\,dt + \mu(t,x)Q(t,x)\,dx = 0$$

be exact. Then μ is called an *integrating factor*. Unfortunately, to find integrating factors is difficult, except in the two following cases:

(1) The quotient

$$a(t) = \frac{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial t}}{Q}$$

is independent of x. Then

$$\mu(t) = e^{\int a(t)\,dt}$$

is an integrating factor.

(2) The quotient

$$b(x) = \frac{\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x}}{P}$$

is independent of t. Then

$$\mu(x) = e^{\int b(x)\,dx}$$

is an integrating factor.

Example 2.6. The equation

$$(t^2 + x^2)\,dt - 2tx\,dx = 0 \tag{2.10}$$

is not exact, since $\partial P/\partial x = 2x \neq -2x = \partial Q/\partial t$. To find an integrating factor we consider the two quotients:

$$\frac{\partial Q}{\partial t} - \frac{\frac{\partial P}{\partial x}}{P} = \frac{-4x}{t^2 + x^2},$$

$$\frac{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial t}}{Q} = \frac{4x}{-2tx} = -\frac{2}{t}, \text{ independent of } x.$$

Hence

$$\mu(t) = e^{-\int 2/t} = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2}$$

is an integrating factor. We multiply equation (2.10) by μ, transforming the ODE in an equivalent one

$$\frac{t^2 + x^2}{t^2}\,dt + \left(-\frac{2x}{t}\right)\,dx = 0,$$

which is exact, since
$$\frac{\partial P}{\partial x} = \frac{2x}{t^2} = \frac{\partial Q}{\partial t}.$$
Now we compute V using (2.4)
$$\frac{\partial V}{\partial t} = P = \frac{t^2 + x^2}{t^2} = 1 + x^2 t^{-2},$$
hence
$$V(t,x) = \int (1 + x^2 t^{-2})\, dt = t - x^2 t^{-1} + \psi(x).$$
Deriving with respect to x we have
$$\frac{\partial V}{\partial x} = -2xt^{-1} + \psi'(x).$$
On the other hand, by (2.5)
$$\frac{\partial V}{\partial x} = Q = -2\frac{x}{t}.$$
We obtain that $\psi = 0$, and the solution is given by
$$t - x(t)^2 t^{-1} = C.$$

2.3. **Linear equations.**

Definition 2.7. The first order ODE
$$\dot{x}(t) + a(t)x(t) = b(t)$$
is called *linear*. Here, $a(t)$ and $b(t)$ are given functions.

To solve the linear equation we proceed as follows. Let $\mu(t) = e^{\int a(t)\, dt}$ and multiply the equation by $\mu(t)$ so that
$$(\dot{x} + a(t)x)\mu(t) = b(t)\mu(t).$$
Notice that $\dot{\mu}(t) = a(t)\mu(t)$ thus,
$$(\dot{x}(t) + a(t)x(t))\mu(t) = \dot{x}(t)\mu(t) + x(t)a(t)\mu(t) = \dot{x}(t)\mu(t) + x(t)\dot{\mu}(t) = \frac{d}{dt}(x(t)\mu(t)).$$
Hence integrating
$$\int \frac{d}{dt}(x(t)\mu(t))\, dt = \int b(t)\mu(t)\, dt \quad \Rightarrow \quad x(t)\mu(t) = \int b(t)\mu(t)\, dt.$$
Solving for $x(t)$ we find
$$x(t) = \frac{1}{\mu(t)} \int b(t)\mu(t)\, dt. \tag{2.11}$$
Recall that the integral symbol means a primitive plus an arbitrary constant. Since for any constant t_0
$$\int_{t_0}^{t} b(s)\mu(s)\, ds$$

is a primitive of $b(t)\mu(t)$, we can write

$$x(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} b(s)\mu(s)\, ds + C \right),$$

from which we can identify the constant C if we look for the solution satisfying $x(t_0) = x_0$:

$$x_0 = \frac{1}{\mu(t_0)} C \Rightarrow C = x_0 \mu(t_0),$$

$$x(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^{t} b(s)\mu(s)\, ds + x_0 \mu(t_0) \right). \tag{2.12}$$

Hence we have proved the following result.

Theorem 2.8. *The unique solution of* $\dot{x}(t) + a(t)x(t) = b(t)$ *passing through* (t_0, x_0) *is given by* (2.12).

Of course, it is not needed to remember the formula. We only need to understand the method used to find it.

Example 2.9. Solve the Cauchy problem $t^2\dot{x} + tx = 1$, $t > 0$, $x(1) = 2$.

Solution: First divide by the coefficient of $\dot{x}$ to have the standard form of the ODE

$$\dot{x} + \frac{x}{t} = \frac{1}{t^2}.$$

We identify here $a(t) = -1/t$ and $b(t) = 1/t^2$. Since $\int a(t)\, dt = \ln t$, we have $\mu(t) = t$. Using (2.11) we get

$$x(t) = \frac{1}{t} \int \frac{1}{t^2} t\, dt = \frac{1}{t} \int \frac{1}{t}\, dt = \frac{1}{t}(\ln t + C).$$

This is the general solution. The individual solution passing through $(1, 2)$ gives $2 = C$, hence $x(t) = \frac{1}{t}(\ln t + 2)$.

Example 2.10. Solve the linear equation $\dot{x} + ax = b$ with initial value $x(t_0) = x_0$, where $a \neq 0$ and b are constants.

Solution: Here $\mu(t) = e^{\int a\, dt} = e^{at}$ and from (2.11)

$$x(t) = e^{-at} \int b e^{at}\, dt = e^{-at} \left(\frac{b}{a} e^{at} + C \right) = \left(\frac{b}{a} + C e^{-at} \right). \tag{2.13}$$

Imposing $x(t_0) = x_0$ it is possible to determine the constant C as follows

$$x_0 = \left(\frac{b}{a} + C e^{-at_0} \right) \Rightarrow C = \left(x_0 - \frac{b}{a} \right) e^{at_0}$$

and plugging this value of C into the (2.13)

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a} \right) e^{-a(t-t_0)}.$$

For instance, the solution of the equation $\dot{x} + 2x = 10$ with $x(0) = -1$ is

$$x(t) = 5 - 6e^{-2t}.$$

2.4. **Phase diagrams for first order scalar equations.** The *phase diagram* of the *autonomous equation* $\dot{x} = f(x)$ consists in a drawing of the graph of function f in the plane $(x, \dot{x})$. The zeroes of f correspond to *steady states*, *stationary points* or *equilibrium points* of the equation, that is, constant solutions of the autonomous ODE.

Definition 2.11 (Stationary points)**.** A stationary point of the autonomous ODE $\dot{x} = f(x)$ is any constant x^0 satisfying $f(x^0) = 0$.

Stationary points are important in the study of the behavior of the dynamics. Analyzing the graph of f, one obtains information on whether the solutions are increasing or decreasing. If $f > 0$ in an interval, then $x(t)$ increases in this interval, which can be indicated by an arrow of motion pointing to the right. Similarly, if $f < 0$, then $x(t)$ decreases in this interval and the arrow that describes the motion of x points to the left.

For scalar ODEs, the sign of the f near a stationary point (if any) gives important information on the behavior of the solution near that point. For systems the situation is more complicated, and will be explored in next sections.

For now, we center on the *scalar case*, $f : D \longrightarrow \mathbb{R}$.

As remarked above, a stationary point is a solution of the ODE, hence if we know that some uniqueness of solutions criterium holds, then no solution can cross through x^0. In the scalar case we have the following, where we are assuming that the stationary point x^0 is isolated:

- $f > 0$ on (a, x^0) and $f > 0$ on (x^0, b). The solution x converges to x^0 from initial conditions $a < x_0 < x^0$ and diverges of x^0 from $b > x_0 > x^0$ (**unstable solution**);
- $f > 0$ on (a, x^0) and $f < 0$ on (x^0, b). The solution x converges to x^0 from every initial condition $a < x_0 < b$ (**stable solution**);
- $f < 0$ on (a, x^0) and $f > 0$ on (x^0, b). The solution x diverges of x^0 from every initial condition $a < x_0 < b$ (**unstable solution**);
- $f < 0$ on (a, x^0) and $f < 0$ on (x^0, b). The solution x diverges of x^0 from initial conditions $a < x_0 < x^0$ and converges to x^0 from $b > x_0 > x^0$ (**unstable solution**).

We can resume the above in the following: a stationary state x^0 is *locally asymptotically stable* if and only if there exists $\delta > 0$ such that for all $x \in (x^0 - \delta, x^0 + \delta)$, $x \neq x^0$ we have

$$(x - x^0)f(x) < 0$$

and it is *unstable* in the other case:

$$(x - x^0)f(x) < 0.$$

Example 2.12. The ODE $\dot{x} = f(x) = x^3 - 2x^2 - 5x + 6$ has three equilibrium points $f(x) = 0$: $x_1^0 = -2$, $x_2^0 = 1$ and $x_0^3 = 3$. The function in negative in $(-\infty, -2)$, positive in $(-2, 1)$, negative in $(1, 3)$ and positive in $(3, \infty)$. Hence,

$$\begin{aligned} &x_1^0 = -2 \text{ is unstable;} \\ &x_2^0 = 1 \text{ is locally asymptotically stable;} \\ &x_0^3 = 3 \text{ is unstable.} \end{aligned}$$

3. Applications

Example 3.1 (Walras adjustment mechanism)**.** Economic models often analyze rates of change of economic variables. In equilibrium analysis the rate of change of the market price for commodity x depends on excess demand E (demand quantity minus the supply quantity, $E = D - S$)

$$\dot{p}(t) = E(p(t)), \tag{3.1}$$

where p is the price. This is a first order differential equation, called the *Walrasian price adjustment mechanism*. Note that $E(p) > 0$ implies that p rises, and $E(p) < 0$ that p falls. Suppose that $D(p) = b - ap$ and $S(p) = \beta + \alpha p$, with $a, b, \alpha, \beta > 0$, with $b > \beta$; then

$$\dot{p} = b - \beta - (a + \alpha)p.$$

This is a linear ODE with constant coefficients. The solution is

$$\begin{aligned} p(t) &= p(0)e^{-(a+\alpha)t} + \frac{b-\beta}{a+\alpha}(1 - e^{-(a+\alpha)t}) \\ &= \left(p(0) - \frac{b-\beta}{a+\alpha}\right) e^{-(a+\alpha)t} + \frac{b-\beta}{\alpha+a}. \end{aligned}$$

The solution tends to the equilibrium solution $p^0 = \frac{b-\beta}{\alpha+a} > 0$.

Example 3.2 (An asset pricing model)**.** Let $p(t)$ denote the price of an equity that pays dividend $D(t)\,dt$, and let r denote the yield on a risk free bond. Consider an interval of time $[t, \tau]$. The total cash flow of the asset in interval $[t, \tau]$ is $\int_t^\tau D(s)\,ds$, and the capital gain in p is $p(\tau) - p(t)$. By a non–arbitrage condition, the cash flow plus capital gains must be equal to earnings of keeping the asset in the bank account, hence

$$\int_t^\tau D(s)\,ds + p(\tau) - p(t) = p(t)e^{r(\tau-t)} - p(t).$$

Dividing by $(\tau - t)$, taking limits as $\tau \to t$, assuming D is (right) continuous and applying L'Hospital rule, we have

$$\lim_{\tau\to t} \frac{\int_t^\tau D(s)\,ds}{\tau - t} = \frac{0}{0} = \lim_{\tau\to t} \frac{D(\tau)}{1} = D(t), \qquad \text{(L'Hospital rule).}$$

$$\lim_{\tau\to t} \frac{p(\tau)-p(t)}{\tau - t} = \dot{p}(t), \qquad \text{(Definition of derivative).}$$

$$\lim_{\tau\to t} \frac{e^{r(\tau-t)}-1}{\tau - t} = \frac{0}{0} = \lim_{\tau\to t} \frac{re^{r(\tau-t)}}{1} = r, \qquad \text{(L'Hospital rule).}$$

Then we get the linear ODE

$$D(t) + \dot{p}(t) = rp(t) \quad \Rightarrow \quad \dot{p}(t) - rp(t) = -D(t), \tag{3.2}$$

which is *the fundamental pricing equation.*

Given dividends $D(t)$, the price of the asset is driven by ODE (3.2). It is a linear equation that can be solved using (2.12) with $\mu(t) = e^{-\int r\,dt} = e^{-rt}$ to give

$$p(t) = e^{rt}\left(\int_0^t -D(s)e^{-rs}\,ds + p(0)\right).$$

Here, $p(0)$ is the current price of the asset, and solving for it we find

$$p(0) = e^{-rt}p(t) + \int_0^t D(s)e^{-rs}\,ds.$$

Notice that we find that the price of the equity at time 0 equals the present value of future dividends only if

$$\lim_{t\to\infty} e^{-rt}p(t) = 0.$$

Supposing that this holds (*the non–bubble condition*), then price of the asset today is

$$p(0) = \int_0^\infty D(s)e^{-rs}\,ds,$$

that is, the *fundamental value of the equity* equals the discounted sum of all future dividends from $t = 0$ onwards.

Some examples: Which is the price of an asset that pays the constant amount of $1\,dt$ euros perpetually? It is

$$p(0) = \int_0^\infty e^{-rs}\,ds = \frac{1}{r}\lim_{t\to\infty}(1 - e^{-rt}) = \frac{1}{r}.$$

Which is the price of an asset that pays $1\,dt$ euro up to $t < 10$ and then $2\,dt$ euros forever if the risk–free rate is $r = 0.025$? It is

$$\begin{aligned} p(0) &= \int_0^{10} e^{-0.025s}\,ds + 2\int_{10}^\infty e^{-0.025s}\,ds \\ &= 40(1 - e^{-0.25}) + 80\lim_{t\to\infty}(e^{-0.25} - e^{-0.025t}) \\ &= 40(1 - e^{-0.25}) + 80e^{-0.25} = 40(1 + e^{-0.25}) = 71.152 \quad \text{euros.} \end{aligned}$$

Example 3.3 (Malthus' model)**.** The British economist Thomas Malthus (1766–1834) observed that many biological populations increase at a rate proportional to the population, P, that is,

$$\dot{P}(t) = rP(t), \tag{3.3}$$

where the constant of proportionality r is called the rate of growth ($r > 0$) or decline ($r < 0$). The mathematical model with $r > 0$ predicts that the population will grow exponentially for all time. Malthus was led to this formulation by inspecting the census records of the United States, which showed a doubling of population every 50 years. Since the means of subsistence were found to increase in arithmetic progression, he argued that the earth could not feed the human population. This point of view had a major impact on social philosophy in the 19th Century. It is immediate to see that the solution is

$$P(t) = P(0)e^{rt},$$

which is unique given the initial condition $P(0)$. The solution shows exponential growth $r > 0$.

As an example, consider the population of the United States in 1800, that was recorded as 5.3 million. Taking $r = 0.03$ (which is a good approximation of the true rate of growth for years around 1800) we get $P(t) = 5.3e^{0.03t}$ million for the population in year $1800+t$. For 1850 it predicts $P(50) = 23.75$ million, whereas the actual population was 23.19. However, for 1900 it gives $P(100) = 106.45$, but the actual population was 76.21. The model approximate the data for years near the initial one, but the accuracy of the approximation diminishes over time because the increase in population is not proportional to the population.

Example 3.4 (Verhulst' model)**.** The Belgian mathematician P.F. Verhulst (1804–1849) observed that limitations on space, food supply or other resources will reduce the growth rate, precluding exponential growth. He modified Eq. (3.3) replacing the constant r by a function $r(P)$

$$\dot{P}(t) = r(P(t))P(t).$$

Verhulst supposed that $r(P) = r - mP$, where r and m are constants. Then, the ODE is

$$\dot{P}(t) = rP(t) - mP^2(t), \tag{3.4}$$

that is also known as the *logistic equation*. It can be rewritten as

$$\dot{P}(t) = r\left(1 - \frac{P(t)}{K}\right)P(t),$$

with $K = r/m$. The constant r is called the intrinsic growth rate, and K is the saturation level or environmental carrying capacity.

The logistic equation is separable and can be integrated explicitly from the identity

$$\int \frac{dP}{P(1 - \frac{P}{K})} = \int r\,dt.$$

Noticing that

$$\frac{1}{P(1 - \frac{P}{K})} = \frac{A}{P} + \frac{B}{1 - \frac{P}{K}}$$

gives $A = 1$ and $B = 1/K$ we have

$$\ln\left|\frac{P}{K-P}\right| = rt + C. \tag{3.5}$$

Imposing that $P(0) = P_0$ is the initial population, the constant C is given by

$$C = \ln\left|\frac{P_0}{K-P_0}\right|.$$

Plugging this value of C into (3.5) and taking the exponential on both sides we get

$$\left|\frac{P}{K-P}\right| = e^{rt}\left|\frac{P_0}{K-P_0}\right|.$$

It is possible to show that if $P_0 < K$, then $P(t) < K$ and that if $P_0 > K$, then $P(t) > K$ for all t, hence eliminating the absolute value on both sides and solving for $P(t)$ we get

$$P(t) = \frac{KP_0}{P(0) + (K - P_0)e^{-rt}}.$$

Notice that $\lim_{t\to\infty} P(t) = K$ if $r > 0$.

Turning back to the example above about United States population, suppose that $K = 300$ (this is close to the actual population of year 2009, and thus a very modest level for the carrying capacity) and $r = 0.03$ (a good estimation of the intrinsic growth rate around 1800, but far away from the actual intrinsic growth rate in year 2009). Recall that the initial data from year 1800 was 5.3. Using this we find $P(50) = 22.38$ and $P(100) = 79.61$, whereas the actual population in 1900 was 76.21.

In Figure 1 it is represented P/K for a population model driven by the logistic equation with $r = 0.71$ (for illustrative purposes), for several initial conditions. The thicker line is the solution with $P(0) = 0.25K$.

Example 3.5 (Population with a threshold)**.** Suppose now that when the population of a species falls below a certain level, the species cannot sustain itself, but otherwise, the population follows logistic growth. To describe this situation, we can consider the differential equation (we omit t)

$$\dot{P} = -r\left(1 - \frac{P}{A}\right)\left(1 - \frac{P}{B}\right)P, \tag{3.6}$$

where $0 < A < B$. The constant A is called the threshold of the population (we will see why afterwards) and B is now the carrying capacity. There are three stationary points, $P = 0$, $P = A$ and $P = B$, corresponding to the equilibrium solutions $P_1(t) = 0$, $P_2(t) = A$ and $P_3(t) = B$, respectively. From Figure 2, it is clear that $P' > 0$ for $A < P < B$. The reverse is true for $y < A$ or $y > B$. Consequently, the equilibrium solution $P_1(t)$ and $P_3(t)$ are asymptotically stable, and the solution $P_2(t)$ is unstable. Notice that the population goes to extinction when $P(0) < A$, so we call A the threshold of the population.

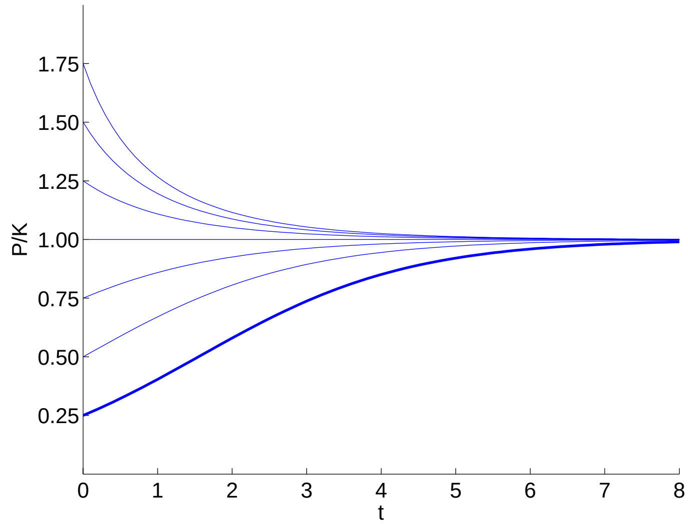


FIGURE 1. P/K versus t for the logistic equation with $r = 0.71$.

In Figure 3 we graph several solutions to $\dot{P} = -0.25P(1-P)(1-P/3)$ using different values for the initial population P_0.When $0 < P(0) < 1$, $\lim_{t\to\infty} P(t) = 0$, and if $1 < P(0) < 3$ or $P(0) > 3$, then $\lim_{t\to\infty} P(t) = 3$.

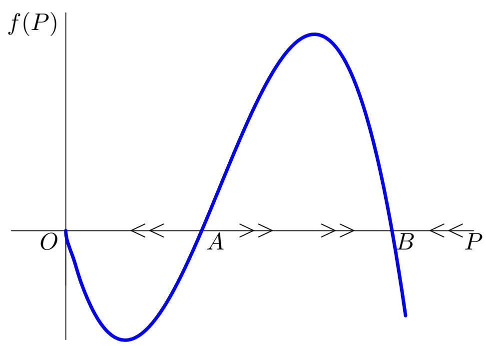


FIGURE 2. Phase space in the population model with threshold.

Example 3.6 (The Solow model)**.** The dynamic economic model of Solow (1956) marked the beginning of modern growth theory. It is based on the following assumptions.

(1) Labour, L, growths at a constant rate n, i.e. $\dot{L}/L = n$;
(2) All saving $S = sY$ are invested in capital formation, $I = \dot{K} + \delta K$, where Y denotes income, K capital and $\delta, s \in (0, 1]$ (δ is capital depreciation):

$$sY = \dot{K} + \delta K.$$

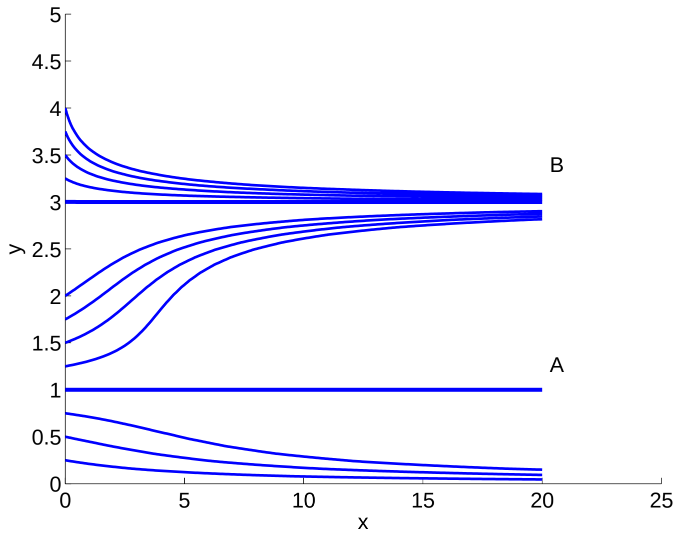


FIGURE 3. Several solutions in the population model with threshold.

(3) The production function $F(L,K)$ depends on labor L and capital K, and shows constant returns, $F(\lambda L, \lambda K) = \lambda F(K,L)$. A typical example is the Coob–Douglas production function $F(K,L) = AL^{\alpha}K^{1-\alpha}$, $\alpha \in [0,1]$. Observe that taking $\lambda = L$

$$Y = F(K,L) = F\left(L\frac{K}{L}, L\right) = LF\left(\frac{K}{L}, 1\right) = Lf(k),$$

where $k = \frac{K}{L}$ and $f(k) = F\left(\frac{K}{L}, 1\right)$.

The fundamental dynamic equation of this growth model is obtained as follows:

$$\begin{aligned}\dot{k} &= \frac{d}{dt}\frac{K}{L} = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - \frac{K}{L}\frac{\dot{L}}{L} \\ &= \frac{sLf(k) - \delta K}{L} - kn = sf(k) - (\delta + n)k.\end{aligned}$$

Thus, we have obtained that the per–capita capital moves according to the ODE

$$\dot{k} = sf(k) - (\delta + n)k.$$

Common assumptions are that the function f is increasing and strictly concave and that there exists a maximal productive stock of capital, k_m, that is, $f(k) < k$ for $k > k_m$ and $f(k) < k$ for $k < k_m$.

There are two steady states $\dot{k} = 0$, $k = 0$ and $k = k_e$ satisfying $sf(k_e) = (\delta + n)k_e$. The steady state 0 is unstable and k_e is stable. This can be seen in the figure below ($\lambda = \delta + n$).

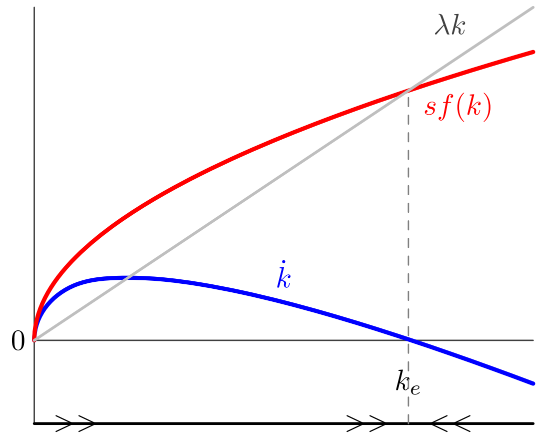


FIGURE 4. Phase diagram in the model of Solow.

4. Second order linear ODEs

Definition 4.1. A second order linear ODEs is of the form

$$\ddot{x} + a_1(t)\dot{x} + a_0(t)x = b(t), \tag{4.1}$$

where a_1, a_0 and b are given functions. In the case that a_1 and a_0 are constant, then the ODE is called of constant coefficients (even if b is not constant). In the case $b = 0$, the ODE is called homogeneous.

Definition 4.2. The general solution of (4.1) is the set of all its solutions; a particular solution is any element of this set.

The space of solutions of the homogeneous ODE has the structure of a vector subspace.

Proposition 4.3. *If x_1 and x_2 are solutions of the homogeneous ODE, then for any constants C_1, C_2, $x(t) = C_1x_1(t) + C_2x_2(t)$ is also a solution.*

Theorem 4.4. *The general solution of the complete ODE* (4.1) *is the sum of the general solution of the homogeneous equation, x_h, and a particular solution, x_p:*

$$x(t) = x_h(t) + x_p(t).$$

Next, we give a result that shows how to find the general solution x_h for the equation with *constant coefficients*

$$\ddot{x} + a_1\dot{x} + a_0x = 0, \qquad a_1, a_0 \quad \text{constant.} \tag{4.2}$$

Definition 4.5. The characteristic equation of (4.2) is

$$r^2 + a_1r + a_0 = 0.$$

Theorem 4.6. *Let r_1, r_2 be the solutions (real or complex) of the characteristic equation. Then, the general solution of the homogeneous equation is of one of the following forms:*

(1) r_1 and r_2 are both real and distinct,

$$x_h(t) = C_1e^{r_1t} + C_2e^{r_2t}.$$

(2) $r_1 = r_2 = r$ *is real and of multiplicity two,*

$$x_h(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

(3) r_1*,* r_2 *are complex conjugates,* $r_{1,2} = a \pm ib$*,*

$$x_h(t) = e^{at}(C_1 \cos bt + C_2 \sin bt).$$

Example 4.7. Find the general solution of the following homogeneous equations.

(1) $\ddot{x} - x = 0$; since $r^2 - 1 = 0$ is the characteristic equation,

$$x_h(t) = C_1 e^t + C_2 e^{-t}.$$

(2) $\ddot{x} - 4\dot{x} + 4x = 0$; since $r^2 - 4r - 4 = (r-2)^2 = 0$ is the characteristic equation,

$$x_h(t) = C_1 e^{2t} + C_2 t e^{2t}.$$

(3) $\ddot{x} + x = 0$; since $r^2 + 1 = 0$ is the characteristic equation, that has roots $\pm i$ ($a = 0$, $b = 1$),

$$x_h(t) = C_1 \cos t + C_2 \sin t.$$

Now, to obtain the general solution of the complete equation, we need to give methods to obtain particular solutions. This is possible only in some limited cases that are described next. Hence, consider the equation with constant coefficients

$$\ddot{x} + a_1 \ddot{x} + a_0 x = b(t),$$

where $b(t)$ is:

-: *A polynomial* $P(t) = b_n t^n + \cdots b_1 t + b_0$ of degree $n = 0, 1, \ldots$;
-: *An exponential* be^{at};
-: *Trigonometric* $b_1 \cos at + b_2 \sin at$;
-: Product and sums of the above (e.g. $(t^2 - t + 1)e^{-t} + 2\sin t$).

Then, a particular solution of the complete equation is of the following corresponding form

-: $x_p(t) = t^s(B_n t^n + \cdots B_1 t + B_0)$, where $s = 2$ if 0 is a double root of the characteristic equation, $s = 1$ if it is simple, and $s = 0$ if it is not a root;
-: $x_p(t) = Bt^s e^{at}$, where $s = 2$ if a is a double root of the characteristic equation, $s = 1$ if it is simple, and $s = 0$ if it is not a root;
-: $x_p(t) = t^s(B_1 \cos at + B_2 \sin at)$, where $s = 1$ if ai is a (complex) root of the characteristic equation and $s = 0$ if it is not (note that ai cannot be a double root of a polynomial of order two and real coefficients);
-: Product and sums of the above (e.g. $t^s(B_2 t^2 + B_1 t + B_0)e^{-t} + t^{s'}(D_1 \sin t + D_2 \cos t)$), where s and s' are determined by the same rules as above.

For instance, the equation $\ddot{x} - 4x = e^{2t}(t+1)$ has characteristic roots 2 and -2. The particular solution is thus of the form $x_p(t) = te^{2t}(B_1 t + B_2)$. However, for the equation with the same independent term $\ddot{x} - 4\dot{x} + 4x = e^{2t}(t+1)$, the particular solution of the form $x_p(t) = t^2 e^{2t}(B_1 t + B_2)$, since 2 is a double root.

The procedure to find x_p is to substitute the guessed form for x_p (depending of the structure of $b(t)$) on the equation, and then to match coefficients to obtain a linear systems for the unknown constants $B_0, \dots B_n$.

Example 4.8. Find particular solutions of the following equations.

(1) $\ddot{x} - x = 2e^{t/2} + e^{-t/2}$. We guess

$$x_p(t) = B_1 e^{t/2} + B_2 e^{-t/2}$$

and put into the equation (after obtaining $\dot{x}_p$ and $\ddot{x}_p$) to have

$$\frac{B_1}{4}e^{t/2} + \frac{B_2}{4}e^{-t/2} - B_1 e^{t/2} - B_2 e^{-t/2} = 2e^{t/2} + e^{-t/2}.$$

Then, $B_1 = -8/3$ and $B_2 = -4/3$.

(2) $\ddot{x} - 4\dot{x} + 4x = t^2 - t$. We guess

$$x_p(t) = B_2 t^2 + B_1 t + B_0.$$

We find $\dot{x}_p = 2B_2 t + B_1$ and $\ddot{x}_p = 2B_2$ and substituting into the equation we obtain

$$2B_2 - 8B_2 t - 4B_1 + 4B_2 t^2 + 4B_1 t + 4B_0 = t^2 - t.$$

Hence, it must be

$$\left.\begin{aligned} 4B_2 &= 1 \\ -8B_2 + 4B_1 &= -1 \\ 2B_2 - 4B_1 + 4B_0 &= 0 \end{aligned}\right\}.$$

Solving, we get $B_0 = 1/8$, $B_1 = 1/4$ and $B_2 = 1/4$.

(3) $\ddot{x} + x = te^{-t} - 2$. We guess

$$x_p(t) = (B_1 t + B_0)e^{-t} + C.$$

We find $\dot{x}_p = B_1 e^{-t} - (B_1 t + B_0)e^{-t}$ and $\ddot{x}_p = -B_1 e^{-t} + (B_1 t + B_0)e^{-t} - B_1 e^{-t}$ and substituting into the equation we get

$$-2B_1 e^{-t} + B_0 e^{-t} + B_1 te^{-t} + B_1 te^{-t} + B_0 e^{-t} + C = te^{-t} - 2.$$

Hence,

$$\left.\begin{aligned} C &= -2 \\ 2B_1 &= 1 \\ -2B_1 + 2B_0 &= 0 \end{aligned}\right\}$$

implies $C = -2$, $B_1 = 1/2$ and $B_0 = 1/2$.

(4) $\ddot{x} + \dot{x} = te^{-t} - 2$. The characteristic equation $r^2 + r = 0$ has solutions $r = 0$ and $r = -1$. The general solution of the homogeneous ODE is $C_1 + C_2 e^{-t}$. Guess that a particular solution is of the form

$$x_p(t) = t^s(B_1 t + B_0)e^{-t} + t^{s'}C,$$

with $s = 1$ and $s' = 1$, since 0 is root of multiplicity 1 to the characteristic equation, and in the independent term, $te^{-t} - 2$, both t and -2 are polynomials. Thus,

$$x_p(t) = (B_1 t^2 + B_0 t)e^{-t} + Ct.$$

We have

$$\dot{x}_p = (2B_1 t + B_0)e^{-t} - (B_1 t^2 + B_0 t)e^{-t} + C$$

and

$$\ddot{x}_p = 2B_1 e^{-t} - (2B_1 t + B_0)e^{-t} - (2B_1 t + B_0)e^{-t} + (B_1 t^2 + B_0 t)e^{-t}.$$

Grouping terms

$$\ddot{x}_p + \dot{x}_p = (-2B_1 t + (2B_1 - B_0))e^{-t} + C$$

and equating to $te^{-t} - 2$, we find the values

$$\left.\begin{array}{rl} C & = -2 \\ B_1 & = -\frac{1}{2} \\ B_0 & = 1 \end{array}\right\}$$

Definition 4.9. The Cauchy problem of the equation (4.1) consists in finding a solution satisfying the initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_1.$$

Example 4.10. Solve the Cauchy problem

$$\ddot{x} - 4\dot{x} + 4x = t^2 - t, \qquad x(0) = 1, \quad \dot{x}(0) = 0.$$

As we know from examples above, the general solution of the complete equation is

$$x(t) = C_1 e^{2t} + C_2 t e^{2t} + \frac{1}{4}t^2 + \frac{1}{4}t + \frac{1}{8}.$$

We need to compute the derivative to use the condition imposed in $\dot{x}(0)$.

$$\dot{x}(t) = 2e^{2t}(C_1 + C_2 t) + C_2 e^{2t} + \frac{t}{2} + \frac{1}{4}.$$

Then

$$\left.\begin{array}{ll} x(0) = 1 & = C_1 + \frac{1}{8} \\ \dot{x}(0) = 0 & = 2C_1 + C_2 + \frac{1}{4} \end{array}\right\}.$$

This linear system can be solved to obtain $C_1 = 7/8$ and $C_2 = -2$. The solution of the Cauchy problem is thus

$$x(t) = \left(\frac{7}{8} - 2t\right)e^{2t} + \frac{1}{4}t^2 + \frac{1}{4}t + \frac{1}{8}.$$

5. Systems of first order ODEs

5.1. **Linear systems.** Consider the n–dimensional linear system of constant coefficients

$$\dot{X}(t) = AX(t) + B,$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \qquad B(t) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

The unknowns are the functions $x_1(t),\dots,x_n(t)$. We are interested only in studying the stability properties of the equilibrium points.

Definition 5.1. A constant vector X^0 is an equilibrium point iff $AX^0+B=0$.

To assure that only one equilibrium exists, we will impose in the following the condition

$$|A|\neq 0.$$

Then the equilibrium is given by

$$X^0=-A^{-1}B.$$

Most often, it is better finding X^0 by solving directly the algebraic system than using the inverse matrix.

Remark 5.2. It can be proved that the solution $X(t)$ of $\dot{X}(t)=AX(t)+B$ is stable (asymptotically stable) if and only if the null solution $X_h(t)\equiv 0$ of the homogeneous system $\dot{X}(t)=AX(t)$ is stable (asymptotically stable). In other words, all solutions of the system $\dot{X}(t)=AX(t)+B$ the same stability properties. Therefore, for studying stability it suffcies to study the homogeneous system,

We center on the two dimensional case $n=2$, that is, in systems with two variables of the form

$$\begin{cases}\dot{x}=a_{11}x+a_{12}y+b_1,\\ \dot{y}=a_{21}x+a_{22}y+b_2.\end{cases}\tag{5.1}$$

where

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\neq 0.$$

Let λ_1, λ_2 be the roots (real or complex) of the characteristic polynomial of A, that is, of the equation

$$p_A(\lambda)=|A-\lambda I_2|=0.$$

We have the following cases (the general solution is that of the homogeneous system):

(1) $\lambda_1\neq\lambda_2$ are real (A is diagonalizable). Let $\mathbf{v}_1\in S(\lambda_1)$ and $\mathbf{v}_2\in S(\lambda_2)$ be eigenvectors. Then, the general solution is

$$X(t)=C_1e^{\lambda_1 t}\mathbf{v}_1+C_2e^{\lambda_2 t}\mathbf{v}_2.$$

(2) $\lambda_1=\lambda_2=\lambda$.

  (a) A is diagonalizable. Let $\mathbf{v}_1,\mathbf{v}_2\in S(\lambda)$ two independent eigenvectors. Then, the general solution is

$$X(t)=e^{\lambda t}(C_1\mathbf{v}_1+C_2\mathbf{v}_2).$$

(b) A is not diagonalizable. Let $\mathbf{v} \in S(\lambda)$ the only independent eigenvector corresponding to λ, and let $\mathbf{w}$ another vector satisfying

$$(A - \lambda I_2)\mathbf{w} = \mathbf{v}.$$

Then, the general solution is

$$X(t) = e^{\lambda t}(C_1\mathbf{v} + C_2\mathbf{w} + C_2 t\mathbf{v}).$$

(3) $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, with $\beta \neq 0$. Then, there are non trivial vectors $\mathbf{v}$ and $\mathbf{w}$ such that the general solution is[1]

$$X(t) = e^{\alpha t}C_1(\mathbf{w}\cos\beta t - \mathbf{v}\sin\beta t) + e^{\alpha t}C_2(\mathbf{w}\sin\beta t + \mathbf{v}\cos\beta t).$$

Using this we can deduce the asymptotic behavior of the solutions.

(1) $\lambda_1 \neq \lambda_2$ are real.
- (a) $\lambda_1, \lambda_2 < 0$. The equilibrium is globally asymptotically stable. It is called an *stable node*.
- (b) $\lambda_1 < 0 < \lambda_2$. The equilibrium is unstable, but solutions for which $C_2 = 0$ converges to X^0. We say that X^0 is a *saddle point*. Initial conditions $X_0 = (x_0, y_0)$ from which the corresponding solution converges form the *stable manifold*, and it is given by the eigenspace $S(\lambda_1)$.
- (c) $\lambda_1, \lambda_2 > 0$. The equilibrium is unstable. It is called an *unstable node*.

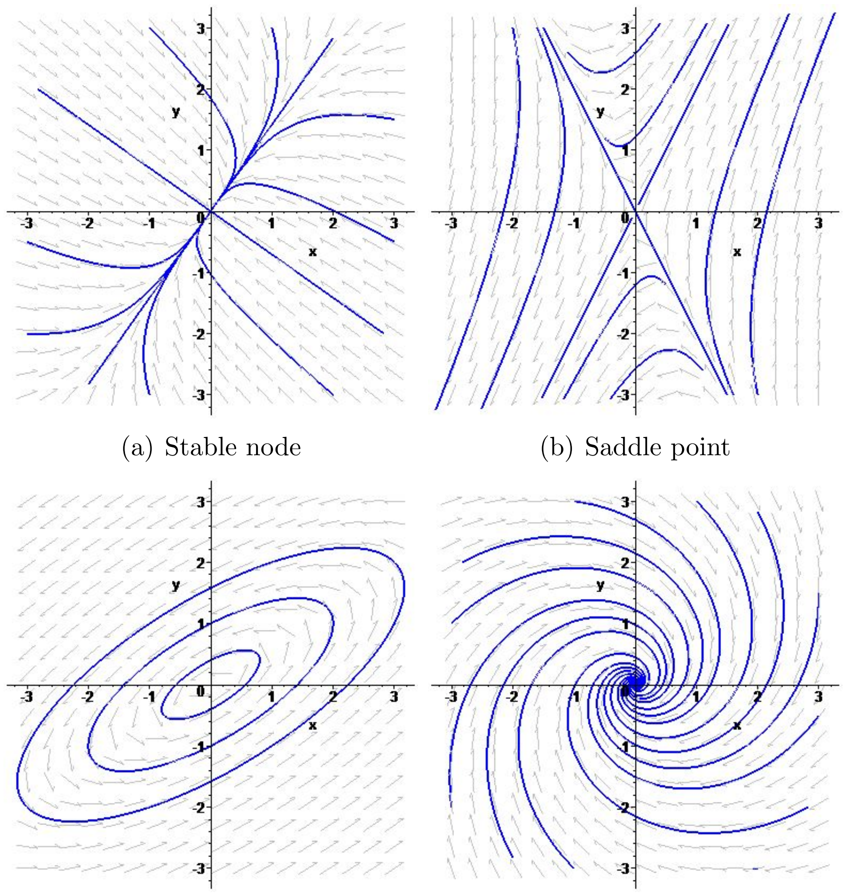
(2) $\lambda_1 = \lambda_2 = \lambda$. The equilibrium is g.a.s. iff $\lambda < 0$. In this case it is called an *improper stable node*. In the case $\lambda > 0$ the system is unstable. In the case $\lambda = 0$, the system is in fact the trivial system $\dot{x} = 0$, $\dot{y} = 0$, which is not interesting.

(3) $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, with $\beta \neq 0$. Notice that the two functions $\mathbf{w}\cos\beta t - \mathbf{v}\sin\beta t$ and $\mathbf{w}\sin\beta t + \mathbf{v}\cos\beta t$ are periodic functions with period $2\pi/\beta$
- (a) The real part $\alpha = 0$. The solution oscillates around X^0 with constant amplitude. It is said that X^0 is a *center*. It is stable, but not g.a.s.
- (b) The real part $\alpha < 0$. The solution oscillates with a decreasing amplitude towards X^0, hence it is g.a.s. and it is called an *spiral point*.

Figure 5 illustrate some of the most important cases. The draws are the phase space of some particular linear systems that show different stability patterns. Continuous lines are solutions $(x(t), y(t))$ of the system, but represented in the xy plane, once the time variable t is eliminated t. Grey small arrows indicate the direction field as determined by the system and point into the direction of displacement of the solutions. The vectors are tangent to the solutions curves in the xy plane.

[1] The vectors can be obtained as solutions to the (complex) linear system $(A - (\alpha + i\beta)I_2)(\mathbf{v} + i\mathbf{w}) = 0$. We are not interested in how to find these vectors, but it can be proved that they satisfy

$$\left((2A - \alpha I_2)^2 + \beta^2 I_2\right)\mathbf{v} = 0, \quad \mathbf{w} = \frac{1}{\beta}(2A - \alpha I_2)\mathbf{v}.$$



(a) Stable node (b) Saddle point

(c) Center (d) Stable spiral

Figure 5. Phase diagram

In the following theorem, note that the assumption $|A| \neq 0$ implies that 0 is not an eigenvalue of A. On the other hand, since the dimension of A is 2, if A has complex eigenvalues, then they have the same real part, as they are conjugate numbers.

Theorem 5.3. *The equilibrium point of the system* (5.1) *is*

(1) Asymptotically stable, if the eigenvalues of A have negative real part;
(2) Stable, but not asymptotically stable, if the eigenvalues of A have null real part;
(3) Unstable. if some of the eigenvalues has positive real part.

Example 5.4. Determine the behavior of solutions near the origin for the system

$$\begin{aligned}\dot{x} &= 3x - 2y,\\ \dot{y} &= 2x - 2y.\end{aligned}$$

Find the general solution.

SOLUTION: The coefficient matrix

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

has characteristic equation

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2,$$

and therefore the eigenvalues are -1 and 2. Therefore the origin is an unstable saddle point.

To find the solution, we need the eigenvectors associated to the eigenvalues, and they are found by solving the homogeneous system

$$\begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\lambda = -1$

$$4v_1 - 2v_2 = 0, \quad 2v_1 - v_2 = 0.$$

and an eigenvector associated to $\lambda_1 = -1$ is $\mathbf{v}_1 = (1, 2)$. When $\lambda = 2$

$$v_1 - 2v_2 = 0, \qquad 2v_1 - 4v_2 = 0,$$

which gives $\mathbf{v}_2 = (2, 1)$. The general solution to the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Example 5.5. Determine the behavior of solutions near the origin for the system

$$\dot{X}(t) = \begin{pmatrix} 3 & b \\ 1 & 1 \end{pmatrix} X(t).$$

SOLUTION: The characteristic equation is

$$\lambda^2 - 4\lambda + (3 - b) = 0.$$

The solutions are

$$\lambda_1 = 2 + \sqrt{1+b}, \qquad \lambda_2 = 2 - \sqrt{1+b}.$$

Note that $b < -1$ implies that λ_1, λ_2 are both complex, with real part $\alpha = 2 > 0$, and that $b \geq -1$ gives $\lambda_1 > 0$ for all b, hence the origin is unstable for all b. However, $\lambda_2 < 0$ for $b > 3$, hence for these values of b the origin is an unstable saddle point.

Example 5.6. Determine the behavior of solutions near the origin for the system

$$\dot{X}(t) = \begin{pmatrix} -a & -1 \\ 1 & -a \end{pmatrix} X(t).$$

SOLUTION: The characteristic equation is

$$\lambda^2 + 2a\lambda + a^2 + 1 = 0,$$

with solutions

$$\lambda_{1,2} = \frac{1}{2}(-2a \pm \sqrt{4a^2 - 4a^2 - 4}) = -a \pm i.$$

The real part is $\alpha = -a$, thus the origin is a g.a.s. spiral for $a > 0$, a center for $a = 0$ and it is an unstable spiral for $a > 0$.

5.2. **Nonlinear systems.** Consider the two dimensional nonlinear system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y). \end{aligned} \tag{5.2}$$

We want to study the stability properties of the equilibrium points of the system. We will use for this a technique that consists in substituting the non–linear system for another that is linear, and that constitutes a local approximation near the equilibrium point of the original system. Then, we will apply, if possible, Theorem 5.3.

The *linearization* of the non–linear system around the equilibrium point (x^0, y^0) is the linear system

$$\begin{aligned} \dot{u} &= \frac{\partial P}{\partial x}(x^0,y^0)\,u + \frac{\partial P}{\partial y}(x^0,y^0)\,v, \\ \dot{v} &= \frac{\partial Q}{\partial x}(x^0,y^0)\,u + \frac{\partial Q}{\partial y}(x^0,y^0)\,v. \end{aligned} \tag{5.3}$$

We call the matrix

$$A(x^0,y^0) = \begin{pmatrix} P_x(x^0,y^0) & P_y(x^0,y^0) \\ Q_x(x^0,y^0) & Q_y(x^0,y^0) \end{pmatrix}$$

the *Jacobian matrix* of the system (5.2).

Example 5.7. The system

$$\begin{aligned} \dot{x} &= y - x, \\ \dot{y} &= -y + \frac{5x^2}{4+x^2} \end{aligned}$$

has three equilibrium points, that is, there are three solutions of the equations

$$\begin{aligned} 0 &= y - x, \\ 0 &= -y + \frac{5x^2}{4+x^2}, \end{aligned}$$

$(0,0)$, $(1,1)$ and $(4,4)$. The linearization of the system about the different equilibrium points can be computed as follows. The partial derivatives are

$$\begin{aligned} P_x(x,y) &= -1, \qquad & P_y(x,y) &= 1, \\ Q_x(x,y) &= \frac{40x}{(x^2+4)^2}, \qquad & Q_y(x,y) &= -1. \end{aligned}$$

Hence, the Jacobian matrices are

$$A(0,0) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad A(1,1) = \begin{pmatrix} -1 & 1 \\ \frac{8}{5} & -1 \end{pmatrix}, \quad A(4,4) = \begin{pmatrix} -1 & 1 \\ \frac{2}{5} & -1 \end{pmatrix},$$

and the associated linear systems are

$$\begin{cases} \dot{u} = -u + v \\ \dot{v} = u - v \end{cases}, \qquad \begin{cases} \dot{u} = -u + v \\ \dot{v} = \dfrac{8}{5}u - v \end{cases}, \qquad \begin{cases} \dot{u} = -u + v \\ \dot{v} = \dfrac{2}{5}u - v \end{cases},$$

respectively. Hence, the linearization depends on the equilibrium point.

Theorem 5.8. *Let* (x^0, y^0) *be an isolated equilibrium point for the nonlinear system* (5.2) *and let* $A = A(x^0, y^0)$ *be the Jacobian matrix for linearization* (5.3)*, with* $|A| \neq 0$*. Then* (x^0, y^0) *is an equilibrium point of the same type as the origin* $(0,0)$ *for the linearization in the following cases.*

(1) The eigenvalues of A *are real, either equal or distinct, and have the same sign (node).*
(2) The eigenvalues of A *are real and have opposite signs (saddle).*
(3) The eigenvalues of A *are complex, but not purely imaginary (spiral)*

Therefore, the exceptional case is when the linearization has a center. The structure for the nonlinear system near the equilibrium points mirrors of the linearization in the non–exceptional cases.

Example 5.9. The nonlinear system

$$\dot{x} = -y - x^3, \quad \dot{y} = x,$$

has Jacobian matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

at the origin $(0,0)$, which has imaginary eigenvalues $\pm i$, and hence $(0,0)$ is a center for the linearization. This is the exceptional case in the theorem, thus we cannot assure that the nonlinear system had a center at $(0,0)$ based in the associated linear system. Actually, it is possible to show that $(0,0)$ is an asymptotically stable spiral for the non–linear system.

Theorem 5.10. *If* $(0,0)$ *is a globally asymptotically stable equilibrium point for* (5.3)*, then it is locally asymptotically stable for* (5.2)*.*

Example 5.11. Consider the nonlinear system

$$\dot{x} = -2x + 3y + xy, \quad \dot{y} = -x + y - 2xy^3,$$

which has an isolated critical point at $(0,0)$. The Jacobian matrix at the origin is

$$A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$$

and it has eigenvalues $-\frac{1}{2} \pm (\frac{\sqrt{3}}{2})i$. Thus the linearization has an asymptotically stable spiral at $(0,0)$, and thus the nonlinear system has an asymptotically stable spiral at $(0,0)$.

Example 5.12. The system

$$\begin{cases} \dot{x} = x(\rho_1 - \kappa_1 x) \\ \dot{y} = y(\rho_2 - \kappa_2 y) \end{cases}$$

models two populations governed by the logistic equation that does not interact to each other. Here ρ_i is the growth rate and ρ_i/κ_i is the saturation level. When both species are present, they compete for a limited amount of available food. To capture the effect of competition, we modify the growth rate factor by $-\alpha_1 y$ and $-\alpha_2 x$ respectively, where α_i is a measure of the effect of one of the species on the other. The system modifies to

$$\begin{cases} \dot{x} = x(\rho_1 - \kappa_1 x - \alpha_1 y) \\ \dot{y} = y(\rho_2 - \kappa_2 y - \alpha_2 x) \end{cases} .$$

Suppose that $\rho_1 = 1$, $\rho_2 = 0.75$, $\kappa_1 = \kappa_2 = 1$, $\alpha_1 = 1$ and $\alpha_2 = 0.5$. There are four equilibrium points: $(0,0)$ (extinction of both species), $(0,0.75)$ (extinction of population x), $(1,0)$ (extinction of population y), and $(0.5,0.5)$ (long–term survival of both species).

Let us study the stability properties of the equilibrium points. The Jacobian matrix of the system is

$$A = A(x,y) = \begin{pmatrix} -2x - y & -x \\ -0.5y & 0.75 - 2y \end{pmatrix}.$$

We analyze the qualitative behavior of the system around the equilibrium points. Recall that $\sigma(A)$ denotes the set of eigenvalues of A.

- $(\mathbf{0},\mathbf{0})$. $\sigma(A) = \{1, 0.75\}$, thus the origin is an unstable node of both the linear and the nonlinear system.
- $(\mathbf{1},\mathbf{0})$. $\sigma(A) = \{-1, 0.25\}$ and it is a saddle point. The stable manifold is the line trough $(1,0)$ in the direction $\mathbf{v}_1 = (1,0)$ and the unstable manifold is generated by $\mathbf{v}_2 = (4,-5)$. Every solution with the initial condition not in the stable manifold depart from $(1,0)$.
- $(\mathbf{0},\mathbf{0.75})$. $\sigma(A) = \{0.25, -0.75\}$, hence again it is a saddle point. The stable manifold is generated by $\mathbf{v}_1 = (0,1)$ and the unstable manifold by $\mathbf{v}_2 = (8,-3)$.
- $(\mathbf{0.5},\mathbf{0.5})$. $\sigma(A) = \{-0.5 \pm \sqrt{2}/4\}$, the eigenvalues have negative real part, thus this is a stable node. All trajectories near $(0.5,0.5)$ converge asymptotically to the equilibrium point of the linear and the nonlinear system.

It is possible to show that for every initial condition with positive population for both species, the dynamical process converges to the coexistence equilibrium.

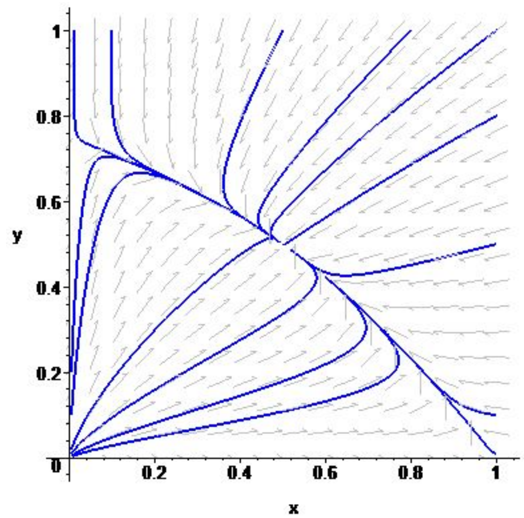


FIGURE 6. Phase portrait, y against x.