TOPICS OF ADVANCED MATHEMATICS FOR ECONOMICS – 2020/2021

Sheet 1. Diagonalization

1-1. *Given the matrix*

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$$

compute its eigenvalues, eigenvectors and diagonalize A.

Solution: The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10$$

The roots are

$$\lambda = \frac{3 \pm \sqrt{9+40}}{2} = -2, 5$$

There are two different eigenvalues. The matrix is diagonalizable.

The eigenspace $S(5)$ is the solution to the system of linear equations

$$\begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which is equivalent to the following one

$$3x - 4y = 0.$$

Therefore, $S(5) < (4,3) >$.

The eigenspace $S(-2)$ is the solution to the system of linear equations

$$\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which is equivalent to the following one

$$x + y = 0.$$

Therefore, $S(-2) =< (1,-1)$.

We conclude that $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} 4 & 1 \\ 3 & -1 \end{pmatrix}$$

1-2. *Given the following matrices*

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

(a) *Compute its eigenvalues, eigenvectors and the eigenspaces.*
(b) *Diagonalize them, whenever possible.*

Solution:

First, we find the eigenvalues of A. The characteristic polynomial is

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & 1-\lambda \end{vmatrix} = -(\lambda+2)(\lambda-1)^2$$

so the eigenvalues ar $\lambda_1 = -2$ with multiplicity $n_1 = 1$ and $\lambda_2 = 1$ with multiplicity $n_2 = 2$.

Now we compute the eigenspace $S(1)$. We solve the following system of linear equations

$$(A - I)\begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 3 & 6 & 0 \\ -3 & -6 & 0 \\ -3 & -6 & 0 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = 0$$

o sea,

$$\begin{aligned} 3x + 6y &= 0 \\ -3x - 6y &= 0 \end{aligned}$$

using y as the parameter, the solution is $S(1) = \{-2y, y, z) : y, z \in \mathbb{R}\} =< (-2, 1, 0), (0, 0, 1) >$ so $\dim S(1) = 2$.

On the other hand, $S(-2)$ is the set of solutions to the system of linear equations

$$(A + 2I) \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 6 & 6 & 0 \\ -3 & -3 & 0 \\ -3 & -6 & 3 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = 0$$

o sea,

$$\begin{aligned} 6x + 6y \quad &= 0 \\ -3x - 3y \quad &= 0 \\ -3x - 6y + z \quad &= 0 \end{aligned}$$

so $S(-2) = \{-z, z, z) : z \in \mathbb{R}\} =< (-1, 1, 1) >$.

The matrix A is diagonalizable and $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of B is

$$|B - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} = -\lambda^2(\lambda - 5)$$

so its eigenvalues are $\lambda_1 = 0$ with multiplicity $n_1 = 2$ and $\lambda_2 = 5$ with multiplicity $n_2 = 1$.

We compute $S(0)$ by solving the linear system of equations

$$(b - 0I) \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} x - 2z \quad &= 0 \\ -2x + 4z \quad &= 0 \end{aligned}$$

We use y and z as parameters. The solution is $x = 2z$. Hence, $S(0) = \{2z, y, z) : y, z \in \mathbb{R}\} =< (2, 0, 1), (0, 1, 0) >$, so $\dim S(0) = 2$.

Now, $S(5)$ is the set of solutions to the system of linear equations

$$(B - 5I) \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} -4 & 0 & -2 \\ 0 & -5 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} -4x - 2z \quad &= 0 \\ -5y \quad &= 0 \\ -2x - z \quad &= 0 \end{aligned}$$

Using x as the parameter, we find that $y = 0$, $z = -2x$. Hence, $S(5) = \{x, 0, -2x) : x \in \mathbb{R}\} =< (1, 0, -2) >$.

The matrix B is diagonalizable and $B = QDQ^{-1}$ with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

The characteristic polynomial of C. is

$$|C - \lambda I| = \begin{vmatrix} 4-\lambda & 5 & -2 \\ -2 & -2-\lambda & 1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = -(\lambda - 1)^3$$

so there is only one eigenvalue $\lambda_1 = 1$ with multiplicity $n_1 = 3$. The space $S(1)$ is the set of solutions to the system of linear equations

$$(C - I) \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} 3 & 5 & -2 \\ -2 & -3 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} 3x+5y-2z &= 0\\ -2x-3y+z &= 0\\ -x-y &= 0 \end{aligned}$$

the solution is $y=-x$, $z=-x$. Using z as the parameter $S(1)=\{-z,z,z):z\in\mathbb{R}\} =< (-1,1,1) >$ so $\dim S(1)=1<n_1=3$ and the matrix C is not diagonalizable.

1-3. *What are the values of a for which the matrix*

$$A=\begin{pmatrix} 1 & 0 & 0\\ a & 1 & 0\\ 1 & 1 & 2 \end{pmatrix}$$

is diagonalizable?

Solution: The characteristic polynomial is

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0\\ a & 1-\lambda & 0\\ 1 & 1 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

There are two eigenvalues $\lambda_1=1$, with multiplicity 2 and $\lambda_2=2$ with multiplicity 1.

The matrix is diagonalizable if and only if $\dim S(1)=2$. The space $S(1)$is the set of solutions to the system of linear equations

$$(A-\lambda I)\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ a & 0 & 0\\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = 0$$

which is the same as

$$\begin{aligned} ax &= 0\\ x+y+z &= 0 \end{aligned}$$

If $a\neq 0$ the solutions are $x=0$, $y=-z$. Hence, $S(1)=\{(0,-z,z):z\in\mathbb{R}\}$ and we see that $\dim S(1)=1$. Hence, if $a\neq 0$ A is not diagonalizable.

But, if $a=0$, the system becomes

$$x+y+z=0$$

so $S(1)=\{(x,y,-x-y):x,y\in\mathbb{R}\}$ and $\dim S(1)=2$. In this case, A is diagonalizable.

1-4. *Show that*

(a) *If A is a diagonalizable matrix, so is A^n for each $n\in\mathbb{N}$.*
(b) *A diagonalizable matrix A is regular if and only if none of its eigenvalues vanishes.*
(c) *If A has an inverse, then both A and A^{-1} have the same eigenvectors and the eigenvalues of A are the reciprocal of the eigenvalues of A^{-1}.*
(d) *A and A^t have the same eigenvalues.*

Solution: Since, $|A^t-\lambda I| = \left|(A-\lambda I)^t\right| = |A-\lambda I|$ the characteristic polynomials of A and A^t are the same. Therefore, the eigenvalues are the same.

1-5. *Study for which values of a and b the matrix* $A=\begin{pmatrix} 5 & 0 & 0\\ 0 & -1 & a\\ 3 & 0 & b \end{pmatrix}$ *is diagonalizable.*

Solution: The characteristic polynomial is

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0\\ 0 & -1-\lambda & a\\ 3 & 0 & b-\lambda \end{vmatrix}| = (5-\lambda)\begin{vmatrix} -1-\lambda & a\\ 0 & b-\lambda \end{vmatrix} = (5-\lambda)(1+\lambda)(b-\lambda)$$

So, the eigenvalues are $\lambda_1=5$, $\lambda_2=-1$ y $\lambda_3=b$. If $b\neq 5$ y $b\neq -1$ there are three different eigenvalues and the matrix is diagonalizable.

If $b=5$ then $\lambda_1=5$ has multiplicity $n_1=2$ and the other eigenvalue has multiplicity 1. The matrix is diagonalizable or not depending on the dimension of $S(5)$. This space is the set of solutions to the system of linear equations

$$(A-5I)\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ 0 & -6 & a\\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix} = 0$$

which is the same as

$$\begin{aligned} -6y + az &= 0 \\ 3x &= 0 \end{aligned}$$

Clearly, $\dim S(5) = 1 < n_1 = 2$, so A is not diagonalizable.

On the other hand, if $b = -1$ the eigenvalues are $\lambda_1 = 5$, with multiplicity $n_1 = 1$ and $\lambda_2 = -1$ with multiplicity $n_2 = 2$. Now The matrix is diagonalizable or not depending on the dimension of $S(-1)$. This space is the set of solutions to the system of linear equations

$$(A+I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} 6x &= 0 \\ az &= 0 \\ 3x &= 0 \end{aligned}$$

and we see that

$$\dim S(-1) = \begin{cases} 1, & \text{si } a \neq 0; \\ 2, & \text{si } a = 0. \end{cases}$$

We could have done this in an easier way, by noting that $\dim S(-1) = 3 - \operatorname{rg}(A+I)$ and

$$\operatorname{rg}(A+I) = \operatorname{rg}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 3 & 0 & 0 \end{pmatrix} = \operatorname{rg}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} 2, & \text{if } a \neq 0; \\ 1, & \text{if } a = 0. \end{cases}$$

Thus, if

$$\begin{cases} b = 5, & \text{then } A \text{ is not diagonalizable;} \\ b = -1, & \text{then } A \text{ is diagonalizable and only if } a = 0; \\ b \neq 5 \text{ y } b \neq -1, & \text{then } A \text{ is diagonalizable.} \end{cases}$$

1-6. *Which of the following matrices are diagonalizable?*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \qquad C\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution: The characteristic polynomial of A is

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda-2)^2$$

so the eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 1$ and $\lambda_2 = 2$ with multiplicity $n_2 = 2$.

The space $S(2)$ is the set of solutions to the system of linear equations

$$(A-2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

that is,

$$\begin{aligned} -x + 2y &= 0 \\ -x + y + z &= 0 \\ y - z &= 0 \end{aligned}$$

The solution is $y = z$, $x = 2y = 2z$ (z is the parameter). Hence, $S(2) = \{(2z, z, z) : z \in \mathbb{R}\} =< (2,1,1) >$ so $\dim S(2) = 1 < n_2 = 2$ and A is not diagonalizable.

The characteristic polynomial of B is

$$|B - \lambda I| = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 1$$

so the eigenvalues are

$$\begin{aligned} \lambda_1 &= \tfrac{-2+\sqrt{4+4}}{2} = -1 + \sqrt{2} \\ \lambda_2 &= \tfrac{-2-\sqrt{4+4}}{2} = -1 - \sqrt{2} \end{aligned}$$

all of the with multiplicity 1. Hence, B is diagonalizable.

The space $S(-1+\sqrt{2})$ is the set of solutions to the system of linear equations

$$\Big(A-(-1+\sqrt{2})I\Big)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix}\begin{pmatrix} x \\ y \\ =0 \end{pmatrix}$$

that is,

$$\begin{aligned} -(1+\sqrt{2})x+y &= 0 \\ x+(1-\sqrt{2})y &= 0 \end{aligned}$$

the solution is $x = y/(1+\sqrt{2})$. Hence, $S(-1+\sqrt{2}) = \{(y/(1+\sqrt{2}), y) : y \in \mathbb{R}\} =< (1/(1+\sqrt{2}), 1) >=< (1, 1+\sqrt{2}) >$. Likewise, $S(-1+\sqrt{2}) = \{(y/(1-\sqrt{2}), y) : y \in \mathbb{R}\} =< (1/(1-\sqrt{2}), 1) >=< (1, 1-\sqrt{2}) >$.

The diagonal form of B is $B = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} -1+\sqrt{2} & 0 \\ 0 & -1-\sqrt{2} \end{pmatrix}$$

Finally, the characteristic polynomial of C is

$$|C-\lambda I| = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (\lambda-1)^2$$

so there is a unique eigenvalue $\lambda = 1$ with multiplicity 2. The eigenspace $S(1)$ is the set of solutions to the system of linear equations

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

so $y = 0$. Hence, $S(1) = \{(x,0) : x \in \mathbb{R}\} =< (1,0) >$ and $\dim S(1) = 1$. Therefore, the C is not diagonalizable.

1-7. *The matrix* $\begin{pmatrix} 1 & 0 & 0 \\ \alpha+1 & 2 & 0 \\ 0 & \alpha+1 & 1 \end{pmatrix}$ *is diagonalizable if and only* α *is...*

Solution: The characteristic polynomial of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \alpha+1 & 2 & 0 \\ 0 & \alpha+1 & 1 \end{pmatrix}$$

is $(\lambda-1)^2(\lambda-2)$. The eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The matrix A is diagonalizable if and only if $\dim S(1) = 2$. The subspace $S(1)$ is the set of solutions to the system of linear equations

$$\left.\begin{aligned} (\alpha+1)x+y &= 0 \\ (\alpha+1)y &= 0 \end{aligned}\right\}$$

If $\alpha \neq -1$ the solution is $x = y = 0$. That is, $S(1) = \{(0,0,z) : z \in \mathbb{R}\}$ and $\dim S(1) = 1$. Therefore, if $\alpha \neq -1$ then A is not diagonalizable.

If $\alpha = -1$ the linear system above reduces to $y = 0$.In this case, $S(1) = \{(x,0,z) : x,z \in \mathbb{R}\}$ and $\dim S(1) = 2$. So, if $\alpha = -1$ the matrix A is diagonalizable.

1-8. *Consider the matrices*

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Find whether they are diagonalizable and, whenever they are, compute their n-th power.

Solution: Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The eigenspaces are $S(1) =< (0,0,1), (-1,1,0) >$ y $S(2) =< (-2,1,0) >$. The matrix A is diagonalizable $A = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

So,

$$A^n = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 0 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1+2^{1+n} & -2+2^{1+n} & 0 \\ 1-2^n & 2-2^n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 0$ with multiplicity $n_1 = 1$ and $\lambda_2 = 2$ with multiplicity $n_2 = 2$. The eigenspaces are $S(0) =< (0,1,1) >$ and $S(2) =< (1,0,1),(1,1,0) >$. The matrix B is diagonalizable: $B = PDP^{-1}$ with

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

so,

$$B^n = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2^n & 0 & 0 \\ 2^{n-1} & 2^{n-1} & -2^{n-1} \\ 2^{n-1} & -2^{n-1} & 2^{n-1} \end{pmatrix}$$

The eigenvalues of

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

are $\lambda_1 = 1$ with multiplicity $n_1 = 2$ and $\lambda_2 = 2$ with multiplicity $n_2 = 1$. The eigenspaces are $S(1) =< (1,0,0),(0,1,0) >$ and $S(2) =< (0,1,1) >$. The matrix C is diagonalizable: $C = PDP^{-1}$ with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

so,

$$C^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2^n-1 \\ 0 & 0 & 2^n \end{pmatrix}$$

1-9. *The following are the characteristic polynomials of some square matrices. Determine which of them correspond to diagonalizable matrices.*

$p(\lambda) = \lambda^2 + 1$ $\quad p(\lambda) = \lambda^2 - 1$
$p(\lambda) = \lambda^2 + \alpha$ $\quad p(\lambda) = \lambda^2 + 2\alpha\lambda + 1$
$p(\lambda) = \lambda^2 + 2\lambda + 1$ $\quad p(\lambda) = (\lambda - 1)^3$
$p(\lambda) = \lambda^3 - 1$

Solution:

1) $p(\lambda) = \lambda^2 + 1$. The matrix is not diagonalizable because not all the roots are real numbers.
2) $p(\lambda) = \lambda^2 + \alpha$. If $\alpha > 0$ the matrix is no diagonalizable because not all the roots are real numbers. If $\alpha < 0$ the characteristic polynomial has two different real roots, so the matrix is diagonalizable. If $\alpha = 0$ there is a unique eigenvalue 0 with multiplicity 2. Hence, either all the entries in the matrix are 0, or else the matrix is no diagonalizable.
3) $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$. We see that -1 is a double root. Therefore, either the matrix is $-I$, or else the matrix is no diagonalizable.
4) $p(\lambda) = \lambda^3 - 1 = (\lambda - 1)(\lambda^2 + \lambda + 1)$ has no real roots. The matrix is no diagonalizable.

5) $p(\lambda) = \lambda^2 - 1$ has two distinct real roots. The matrix is diagonalizable.
6) $p(\lambda) = \lambda^2 + 2\alpha\lambda + 1$. The roots are $\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}$. Thus,

- If $|\alpha| > 1$, the matrix is diagonalizable.
- If $|\alpha| < 1$, the matrix is not diagonalizable.
- If $|\alpha| = 1$, we are in case 3).

1-10. *Determine whether the following matrices are diagonalizable. Compute the n-th power whenever they are diagonalizable.*

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & \alpha \end{pmatrix} \quad B = \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:
1) The matrix A is of order 2 and its unique eigenvalue is α of multiplicity 2. Therefore, A is not diagonalizable.
2) The characteristic polynomial of B is $(\lambda - \alpha)^2 - 1$. The roots are $\alpha \pm 1$ so B is diagonalizable. The eigenvalues are

$$S(\alpha - 1) = <(-1, 1)>, \qquad S(\alpha + 1) = <(1, 1)>$$

and $B = PDP^{-1}$ con

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \alpha + 1 \end{pmatrix}$$

Thus,

$$B^n = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\alpha-1)^n & 0 \\ 0 & (\alpha+1)^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} =$$

$$\frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (\alpha-1)^n & 0 \\ 0 & (\alpha+1)^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\alpha-1)^n + (\alpha+1)^n & -(\alpha-1)^n + (\alpha+1)^n \\ -(\alpha-1)^n + (\alpha+1)^n & (\alpha-1)^n + (\alpha+1)^n \end{pmatrix}$$

3) The eigenvalues of

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

are $\lambda_1 = 1$ with multiplicity $n_1 = 3$. Since,

$$S(1) = <(1, 0, 0)>$$

the matrix is not diagonalizable.

1-11. *Study for what values of the parameters the following matrices are diagonalizable. Find the eigenvalues and eigenvectors.*

$$A = \begin{pmatrix} a & b & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & -2-\alpha \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

1-12. *The matrix*

$$\begin{pmatrix} a & 1 & p \\ b & 2 & q \\ c & -1 & r \end{pmatrix}$$

has $(1, 1, 0)$, $(-1, 0, 2)$ *and* $(0, 1, -1)$ *as eigenvectors. Compute its eigenvalues.*

1-13. *Determine whether the following matrices are diagonalizable. If possible, write their diagonal form.*

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 5 & 7 & 5 \\ -6 & -5 & -3 \\ 4 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad F = \begin{pmatrix} 5 & -10 & 8 \\ -10 & 2 & 2 \\ 8 & 2 & 11 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{pmatrix} \quad H = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix} \quad I = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad K = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix} \quad L = \begin{pmatrix} -9 & 1 & 1 \\ -18 & 0 & 3 \\ -21 & 4 & 0 \end{pmatrix}$$

Solution:

1) The eigenvalues of

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

are $-2, 4, 4$. Also, $S(-2) =< (-1, 1, 1) >$, $S(4) =< (1, -1, 1) >$, so the matrix is diagonalizable.

2) The eigenvalues of

$$B = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

are $-1, -1, -1$. Since, B is not already in diagonal form, it is not diagonalizable.

5) The eigenvalues of

$$E = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 2 & 0 \\ 0 & 3 & -2 \end{pmatrix}$$

are $-2, -1, 2$. Since they are all distinct then E is diagonalizable. Also, $E = PDP^{-1}$ with

$$P = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 12 \\ 1 & 0 & 9 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

12) The eigenvalues of

$$L = \begin{pmatrix} -9 & 1 & 1 \\ -18 & 0 & 3 \\ -21 & 4 & 0 \end{pmatrix}$$

are $-3, -3, -3$. Since L is not already in diagonal form, it is not diagonalizable.