Midterm – March 2017

Exercise 1. Consider the contract design problem of a Principal whose revenue is a random variable taking values $x_1 = 4$ and $x_2 = 8$ with probabilities that depend on the effort an Agent exerts, $e \in [0,1]$; specifically, $p_1(e) = (2-e)/3$, and $p_2(e) = 1 - p_1(e)$. The Agent's reservation utility is $\underline{u} = 1$, and his cost of effort is $v(e) = e/2$.

(a) (20 points) Assuming that effort is verifiable, the Principal is risk-neutral, and the preferences of the Agent are represented by the von Neumann-Morgenstern utility function $u(x) = \sqrt{x}$, determine the optimal contract and calculate the Principal's expected profit. (Hint. For $e \in [0,1]$ identify the wage schedule the Principal's will offer, and then identify the optimal level of effort by solving the Principal's maximization problem.)

(b) (20 points) Assume now that the Principal is risk averse, and the Agent is risk neutral. Normalize the Agent's utility function to $u(x) = x$. Determine the optimal contract and calculate the Principal's profit. Does it matter whether or not effort is verifiable?

(c) (25 points) Now return to the assumptions of part (a), but assume that only the effort levels $e = 0$ and $e = 1$ are feasible, and that effort is *not* verifiable. Determine the optimal contract and calculate the Principal's expected profit.

(a) The optimal contract requiring the Agent exerting effort $e \in [0,1]$ involves paying the Agent a fixed wage $\bar{w}(e)$, which is obtained solving the participation constraint with equality,

$$Eu(\bar{w}(e)) = \underline{u} + v(e) \Leftrightarrow \sqrt{w(e)} = 1 + \frac{e}{2};$$

Hence the optimal contract requiring the Agent exerting effort $e = 1$ involves paying the Agent a fixed wage

$$w(e) = \left(1 + \frac{e}{2}\right)^2.$$

For $e \in [0,1]$ the Principal's profit is

$$E[X(e)] - w(e) = \frac{(2-e)}{3}(4) + \left(1 - \frac{(2-e)}{3}\right)(8) - \left(1 + \frac{e}{2}\right)^2.$$

which is maximize by setting e as the solution to

$$\frac{d}{de}\left(E[X(e)] - w(e)\right) = \frac{1}{3} - \frac{e}{2} = 0.$$

Solving this equation we get $e^* = 2/3$. Hence the optimal contract is

$$(e^*, w(e^*)) = (2/3, 16/9),$$

and the Principal's expected profit is

$$E[X(e^*)] - \bar{w}(e^*) = \frac{(2-2/3)}{3}(4) + \left(1 - \frac{(2-2/3)}{3}\right)(8) - \left(1 + \frac{2/3}{2}\right)^2 = \frac{40}{9}.$$

(b) In this case the optimal contract involves a franchise, i.e., the principal transfers the business to the Agent for a fixed payment y^*, where y^* is the maximum price the Agent is willing to pay. In order to calculate y^* we solve the Agent's maximization problem,

$$\max_{e \in [0,1]} Eu[X(e)] - v(e) - y = E[X(e)] - \frac{e}{2} - y.$$

Since y is a constant in this problem, the solution to this problem e^* is independent of y. Once we identify e^* we obtain y^* as the solution to the equation

$$E[X(e^*)] - \frac{e^*}{2} - y = \underline{u}.$$

It is easy to see that $E[X(e)] - e$ is increasing in $[0,1]$: taking derivative we get

$$\frac{d}{de}\left(E[X(e)] - \frac{e}{2}\right) = \frac{d}{de}\left(\frac{(2-e)}{3}(4) + \left(1 - \frac{(2-e)}{3}\right)(8) - \frac{e}{2}\right) = \frac{5}{6} > 0.$$

Hence $e^* = 1$, and

$$E[X(e^*)] - \frac{e^*}{2} = \frac{(2-1)}{3}(4) + \left(1 - \frac{(2-1)}{3}\right)(8) - \frac{1}{2} = \frac{37}{6}.$$

Therefore y^* is the solution to the equation

$$\frac{37}{6} - y = 1,$$

that is,

$$y^* = \frac{31}{6} \approx 5.16.$$

(c) If effort is not verifiable, then the contract $(0, \underline{W}^*)$*, where* $\underline{W}^* = (\underline{w}_1, \underline{w}_2)$ *and*

$$\underline{w}_1 = \underline{w}_2 = w(0) = \left(1 + \frac{0}{2}\right)^2 = 1,$$

continues to satisfy the participation and incentive constraints. The Principal's profit with this contract is

$$E[X(0)] - w(0) = \frac{(2-0)}{3}(4) + \left(1 - \frac{(2-0)}{3}\right)(8) - 1 = \frac{13}{3}.$$

If the Principal wants the Agent to exert effort $e = 1$*, then the wage contract* $\bar{W}^* = (\bar{w}_1, \bar{w}_2)$ *must satisfy the participation and incentive constraints with equality, that is,*

$$\begin{aligned} Eu(\bar{W}^*(1)) - v(1) &= \underline{u} \\ Eu(\bar{W}^*(1)) - v(1) &= Eu(\bar{W}^*(0)) - v(0). \end{aligned}$$

That is,

$$\begin{aligned} \frac{(2-1)}{3}\sqrt{\bar{w}_1} + \left(1 - \frac{(2-1)}{3}\right)\sqrt{\bar{w}_2} - \frac{1}{2} &= 1 \\ \frac{(2-1)}{3}\sqrt{\bar{w}_1} + \left(1 - \frac{(2-1)}{3}\right)\sqrt{\bar{w}_2} - \frac{1}{2} &= \frac{(2-0)}{3}\sqrt{\bar{w}_1} + \left(1 - \frac{(2-0)}{3}\right)\sqrt{\bar{w}_2} - \frac{0}{2} \end{aligned}$$

Solving this system we get

$$\bar{w}_1 = \frac{1}{4}, \bar{w}_2 = 4.$$

The Principal's profit with this contract is

$$\begin{aligned} E[X(1) - \bar{W}^*(1)] &= \frac{(2-1)}{3}\left(4 - \frac{1}{4}\right) + \left(1 - \frac{(2-1)}{3}\right)(8-4) \\ &= \frac{47}{12} \\ &< \frac{13}{3}. \end{aligned}$$

Hence the optimal contract is $(0, \underline{W}^*)$*, and the Principal's expected profit is* $13/3$.

Exercise 2. A good exist in two qualities, high (H) and low (L). There are four buyers each of whom wants to buy a single unit, and has a value for high quality good equal to $u^H = 1$, and a value for low quality good equal to $u^L = 1/5$. There are two sellers who own each a unit of high quality and have an opportunity cost equal to $c^H = 3/5$, and one other seller with a unit of low quality good and an opportunity cost equal to $c^L = 0$. Buyers and sellers are risk neutral. In your answers to the questions below, please, provide graphs when appropriate and explain the calculations leading the your conclusions.

(a) (10 points) Calculate the competitive equilibrium assuming that the quality of the good is observable.

In questions (b) and (c) assume that the quality of the good is *not* observable.

(b) (10 points) Calculate the competitive equilibrium.

(c) (15 points) Determine the impact on equilibrium and on the welfare of buyers and sellers of a subsidy of $s \in [0, 2/15]$ euros to each buyer of a unit of the good.

(a) If quality is observable the supply of each quality is

$$S^H(p) = \begin{cases} 0 & \text{if } p < 3/5 \\ \{0,1,2\} & \text{if } p = 3/5 \\ 2 & \text{if } p > 3/5 \end{cases}, \quad S^L(p) = \begin{cases} \{0,1\} & \text{if } p = 0 \\ 1 & \text{if } p > 0. \end{cases}$$

For prices (p^H, p^L), the demand of both qualities is (considering only the interesting price vectors):

$$(D^H(p^H,p^L), D^L(p^H,p^L)) = \begin{cases} (0,4) & \text{if } 0 < 1 - p^H < 1/5 - p^L \\ \{(x, 4-x),\ x \in \{0,1,2,3,4\}\} & \text{if } 0 \leq 1 - p^H = 1/5 - p^L \\ (4,0) & \text{if } 1 - p^H > 1/5 - p^L > 0. \end{cases}$$

The competitive equilibrium (CE) price vector (p^H, p^L) must clear both markets. It is easy to see that $(\bar{p}^H, \bar{p}^L) = (1, 1/5)$ is the unique CE. For this price vector $1 - \bar{p}^H = 1/5 - \bar{p}^L = 0$, and therefore buyers are indifferent between buying or not one unit of either quality (we are in the middle case of formula above describing the demand), the sellers of high quality want to sell their unit (that is, $S^H(1) = 2$), and the seller of low quality wants to sell his unit (that is, $S^L(1/5) = 1$). This is clearly the unique CE. In this equilibrium sellers capture all the surplus. (Note that sellers are the "short side" in both markets.)

(b) In this case both qualities are traded in a single market, and the supply is $S(p) = S^H(p) + S^L(p)$*, while the demand is*

$$D(p) = \begin{cases} 4 & \text{if } p \in [0, 1/5) \\ \{0,1,2,3,4\} & \text{if } p = 1/5 \\ 0 & \text{if } p \in [1/5, 3/5] \\ \{0,1,2,3,4\} & \text{if } p = 3/5 \\ 4 & \text{if } p \in [3/5, \bar{u}] \\ \{0,1,2,3,4\} & \text{if } p = \bar{u} \\ 0 & \text{if } p > \bar{u} \end{cases}$$

In order to see this, note that for prices $p \in (1/5, 3/5)$ *only low quality sellers supply, and therefore the value to a buyer of a unit supplied is* $1/5 < p$*, and hence* $D(p) = 0$*, and for* $p \geq 3/5$ *the value to a buyer of a unit supplied is*

$$\bar{u} = \frac{2}{3}(1) + \frac{1}{3}(\frac{1}{5}) = \frac{11}{15} > \frac{9}{15} = \frac{3}{5},$$

and therefore the demand is $D(p) = 4$ *. Therefore there are two CE. For the first one, the price is* $p^* = 1/5$*, and only low quality trades. The CE is inefficient (in the classical sense) because high quality does not trade even though there are gains to trade. In this CE the surplus is capture by low quality sellers.*

In the other CE, the price is $p^* = [3/5, \bar{u}]$ *and both qualities are traded.*

(c) The subsidy s may be interpret as a reduction of the price, or equivalently as an increment of the value of both qualities. Using this second interpretation we can calculate the value of a random unit when both types of sellers supply as

$$\bar{u}(s) = \frac{2}{3}(1) + \frac{1}{3}(\frac{1}{5}) + s = \frac{11}{15} + s.$$

Thus, the demand is

$$D(p,s) = \begin{cases} 4 & \text{if } p \in [0, 1/5 + s) \\ \{0,1,2,3,4\} & \text{if } p = 1/5 + s \\ 0 & \text{if } p \in [1/5 + s, 3/5 + s] \\ \{0,1,2,3,4\} & \text{if } p = 3/5 + s \\ 4 & \text{if } p \in [3/5 + s, \bar{u}] \\ \{0,1,2,3,4\} & \text{if } p = \bar{u} \\ 0 & \text{if } p > \bar{u} \end{cases}$$

CE is analogous to b). The subsidy is captured by the sellers.

Exercise 3. Consider a population of individuals, each of whom has a labor income $W = 5$. For half of the individuals the probability of getting sick, and hence being unable to work, which results in loosing their labor income, is $p^L = 1/4$, whereas for the remaining half this probability is $p^H = 1/2$. Individuals' preferences are described by the von Neumann-Morgenstern utility function $u(x) = \ln x$. This information is common knowledge. There is a competitive insurance market. (In answering the questions below, when appropriate show your solution graphically first, and then do the algebra and discuss your results.)

(a) (10 points) Determine the policies that will be offered assuming that insurance companies have access to a health history database which allows them to know whether any particular individual's probability of getting sick is p^H or p^L. Calculate the welfare of each risk type individual.

(b) (10 points) Suppose that the government passes legislation that forbids insurance companies discriminating individuals based on their health history, and makes it mandatory for every individual to subscribe a full insurance policy. Determine the policies that will be offered in this scenario, and whether an individual will be better off or worse off that in the scenario (a).

(c) (15 points) Now assume that a new president is elected that eliminates the legislation described in part (b), and at the same time forbids access to the health history database, thus making private information each person's type. Verify that a competitive equilibrium exists, and identify the policies that will be offered. Is social welfare larger or smaller than in (a) and (b)?

(a) In a CE insurance companies will in each market (the market for high risk individuals and the market for low risk individuals) the full insurance fair policy. These policies are $\left(I^H, D^H\right) = (5p^H, 0) = (5/2, 0)$ *and* $\left(I^L, D^L\right) = (5p^L, 0) = (5/4, 0)$. *Here one must supply a graph. The expected utility of each type are*

$$Eu^L = \ln(5 - 5/4) = \ln(15/4) = \ln 5 + \ln 3 - 2\ln 2 \simeq 1.32,$$

and

$$Eu^H = \ln(5 - 5/2) = \ln(5/2) = \ln 5 - \ln 2 \simeq 0.91.$$

The way the above calculations have been written makes it obvious that $Eu^L > Eu^H$.

(b) In this case the single policy that will be subscribed in the pooling CE is the full insurance, fair policy $(\bar{I}, \bar{D}) = (5\bar{p}, 0)$, *where*

$$\bar{p} = p^H/2 + p^L/2 = 1/4 + 1/8 = 3/8,$$

and the expected utility of both types of individuals is

$$E\bar{u} = \ln(5 - 5(3/8) = \ln(25/8) = 2\ln 5 - 3\ln 2 \simeq 1.139\,4.$$

Obviously, high risk individuals are better off than in (a), but low risk individual are worse off than in (a) since

$$\begin{aligned} E\bar{u} - Eu^H &= \ln 5 - \ln 4 > 0 \\ E\bar{u} - Eu^L &= \ln 5 - \ln 6 < 0. \end{aligned}$$

This is quite obvious – just checking!

(c) In this case a separating equilibrium, if it exists, will emerge. The separating policies are $\left(\tilde{I}^H, \tilde{D}^H\right) = (5p^H, 0) = (5/2, 0)$ *(identified in (a)), and* $\left(\tilde{I}^L, \tilde{D}^L\right)$ *which is identified by the system of equations*

$$\begin{aligned} \tilde{I}^L &= p^L\left(5 - \tilde{D}^L\right) \\ p^H \ln\left(5 - \tilde{I}^L - \tilde{D}^L\right) + \left(1 - p^H\right)\ln\left(5 - \tilde{I}^L\right) &= \ln(5 - 5/2) \end{aligned}$$

That is $\tilde{D}^L = x$ *must solve the equation*

$$\frac{1}{2}\ln\left(5 - \frac{1}{4}(5 - x) - x\right) + \frac{1}{2}\ln\left(5 - \frac{1}{4}(5 - x)\right) = \ln(5/2).$$

That is,

$$\left(5 - \frac{1}{4}(5 - x) - x\right)\left(5 - \frac{1}{4}(5 - x)\right) = (5/2)^2.$$

Solving this equation we get

$$\tilde{D}^L = \frac{10}{3}\sqrt{2}\sqrt{3} - 5 \simeq 3.165.$$

Hence

$$\tilde{I}^L = \frac{1}{4}\left(5 - \left(\frac{10}{3}\sqrt{2}\sqrt{3} - 5\right)\right) = \frac{5}{6}\left(3 - \sqrt{6}\right) \simeq 0.45$$

The expected utility of the low risk individual with the separating insurance policy is

$$\begin{aligned} E\tilde{u}^L &= \frac{1}{4}\ln\left(5 - \frac{5}{6}\left(3 - \sqrt{6}\right) - \left(\frac{10}{3}\sqrt{2}\sqrt{3} - 5\right)\right) + \frac{3}{4}\ln\left(5 - \frac{5}{6}\left(3 - \sqrt{6}\right)\right) \\ &= 1.215 \\ &> 1.1394 = E\bar{u}. \end{aligned}$$

Therefore the separating policies form a CE.

The social welfare for this CE is:

$$E\tilde{u}^L + u^H \simeq 1.215 + 0.91 = 2.125$$

Compared with part a):

$$u^L + u^H \simeq 1.32 + 0.91 = 2.24$$

and part b):

$$2E\bar{u} \simeq 2 * 1.1394 = 2.279$$